The local integration of Leibniz algebras

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Contents

Introduction

The main result of this thesis is a local answer of the *coquecigrue problem.* By coquecigrue problem, we talk about the problem of integrating Leibniz algebras. This question was formulated by J.-L. Loday in [Lod93] and consists in finding a generalisation of the Lie's third theorem for Leibniz algebras. This theorem establishes that for every Lie algebra $\mathfrak{g}$, there exists a Lie group G such that its tangent space at 1 is provided with a structure of Lie algebra isomorphic to $\mathfrak{g}$. Leibniz algebras are generalisations of Lie algebras, they are their non-commutative analogues. Precisely, a *(left) Leibniz algebra* (over $\mathbb{R}$) is an $\mathbb{R}$-vector space $\mathfrak{g}$ provided with a bilinear form $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the bracket and satisfying the *(left) Leibniz identity* for all x, y and z in $\mathfrak{g}$

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

Hence, a natural question is to know if, for every Leibniz algebra, there exists a manifold provided with an algebraic structure generalizing the group structure, and such that the tangent space in a distinguished point, called 1, can be provided with a Leibniz algebra structure isomorphic to the given Leibniz algebra. As we want this integration to be the generalisation of the Lie algebra case, we also require that, when the Leibniz algebra is a Lie algebra, the integrating manifold is a Lie group.

The main result on this question was given by M.K. Kinyon in [Kin07]. In his article, he solves the particular case of *split* Leibniz algebras, that is Leibniz algebras which are isomorphic to the *demisemidirect product* of a Lie algebra and a module over this Lie algebra. That is, Leibniz algebras which are isomorphic to $\mathfrak{g} \oplus \mathfrak{a}$ as vector space, where the bracket is given by $[(x,a),(y,b)] = ([x,y], x.a)$. In this case, he shows that the algebraic structure which answers the problem is the structure of a *digroup.* A digroup is a set with two binary operation $\vdash$ and $\dashv$, a neutral element 1 and some compatibility conditions. More precisely, he shows that a digroup structure induces a *pointed rack* structure (pointed in 1), and it is this algebraic structure which gives the tangent space at 1 a Leibniz algebra structure. Of course, not every Leibniz algebra is isomorphic to a demisemidirect product, so we have to find a more general structure to solve the problem. One should think that the right structure is that of a pointed rack, but M.K. Kinyon showed in [Kin07] that the second condition (Lie algebra becomes integrated into a Lie group) is not always fulfilled. Thus we have to specify the structure inside the category of pointed racks.

In this thesis we don't give a complete answer to the coquecigrue problem in the sense that we only construct a *local* algebraic structure and not a global one. Indeed, to define an algebraic structure on a tangent space at a given point on a manifold, we just need an algebraic structure in a neighborhood of this point. We will show in chapter 3 that a local answer to the problem is given by the pointed augmented local racks which are abelian extensions of a Lie group by an anti-symmetric module.

Our approach to the problem is similar to one given by E. Cartan in [Car30]. The main idea comes from the fact that we know the Lie's third theorem on a class of Lie algebras. For example, every abelian Lie algebra or every Lie subalgebras of the Lie algebra $End(V)$ is integrable (using the Lie's first theorem). More precisely, let $\mathfrak{g}$ a Lie algebra, $Z(\mathfrak{g})$ its center and $\mathfrak{g}_0$ the quotient of $\mathfrak{g}$ by $Z(\mathfrak{g})$. The Lie algebra $Z(\mathfrak{g})$ abelian and $\mathfrak{g}_0$ is a Lie subalgebra of $End(\mathfrak{g})$, thus there exist Lie groups, respectively $Z(\mathfrak{g})$ and G_0, which integrate these Lie algebras. As a vector space, $\mathfrak{g}$ is isomorphic to the direct sum $\mathfrak{g}_0 \oplus Z(\mathfrak{g})$, thus the tangent space at $(1,0)$ of the manifold $G_0 \times Z(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$. As a Lie algebra, $\mathfrak{g}$ is isomorphic to the central extension $\mathfrak{g}_0 \oplus_\omega Z(\mathfrak{g})$ where ω is a Lie 2-cocycle on $\mathfrak{g}_0$ with coefficients in $Z(\mathfrak{g})$. That is, the bracket on $\mathfrak{g}_0 \oplus_\omega Z(\mathfrak{g})$ is defined by

$$[(x,a),(y,b)] = \big([x,y], \omega(x,y)\big) \tag{1}$$

where ω is an anti-symmetric bilinear form on $\mathfrak{g}_0$ with value on $Z(\mathfrak{g})$ which satisfies the Lie algebra cocycle identity

$$\omega([x,y],z) - \omega(x,[y,z]) + \omega(y,[x,z]) = 0$$

Hence we have to find a group structure on $G_0 \times Z(\mathfrak{g})$ which gives this Lie algebra structure on the tangent space at $(1,0)$. It is clear that the bracket (1) is completely determined by the bracket in $\mathfrak{g}_0$ and the cocycle ω. Hence, the only thing we have to understand is ω. The Lie algebra $\mathfrak{g}$ is a central extension of $\mathfrak{g}_0$ by $Z(\mathfrak{g})$, thus we can hope that the Lie group which integrates $\mathfrak{g}$ should be a central extension of G_0 by $Z(\mathfrak{g})$. To follow this idea, we have to find a group 2-cocycle on G_0 with coefficients in $Z(\mathfrak{g})$. In this case, the group structure on $G_0 \times Z(\mathfrak{g})$ is given by

$$(g,a).(h,b) = \big(gh, a+b+f(g,h)\big) \tag{2}$$

where f is a map from $G \times G \to Z(\mathfrak{g})$ vanishing on $(1,g)$ and $(g,1)$ and satisfying the group cocycle identity

$$f(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0$$

With such a cocycle, the conjugation in the group is given by the formula

$$(g,a).(h,b).(g,a)^{-1} = \big(ghg^{-1}, a + f(g,h) - f(ghg^{-1},g)\big) \tag{3}$$

and by imposing a smoothness condition on f in a neighborhood of 1, then we can differentiate this formula twice, and obtain a bracket on $\mathfrak{g}_0 \oplus Z(\mathfrak{g})$ defined by

$$[(x,a),(y,b)] = \big([x,y], D^2f(x,y)\big)$$

where $D^2f(x,y) = d^2f(1,1)((x,0),(0,y)) - d^2f(1,1)((y,0),(0,x))$. Thus, if $D^2f(x,y)$ equals $\omega(x,y)$, then we recover the bracket (1). Hence, if we associate to ω a group cocyle f satisfying some smoothness conditions and such that $D^2f = \omega$, then our integration problem is solved. This can be done in two steps. The first one consists in finding a local Lie group cocycle defined around 1. Precisely, we want a map f defined on a subset of $G_0 \times G_0$ containing $(1,1)$ with values in $Z(\mathfrak{g})$ which satisfies the local group cocycle identity (cf. [Est54] for a definition of local group). We can construct explicitely such a local group cocycle. This construction is the following one (cf. Lemma 5.2 in [Nee04]) :

Let V be an open convex 0-neighborhood in $\mathfrak{g}_0$ and $\phi : V \to G_0$ a chart of G_0 with $\phi(0) = 1$ and $d\phi(0) = id_{\mathfrak{g}_0}$. For all $(g,h) \in \phi(V) \times \phi(V)$ such that $gh \in \phi(V)$ let's define $f(g,h) \in Z(\mathfrak{g})$ by the formula

$$f(g,h) = \int_{\gamma_{g,h}} \omega^{inv}$$

where $\omega^{inv} \in \Omega^2(G_0, Z(\mathfrak{g}))$ is the invariant differential form on G_0 associated to ω and $\gamma_{g,h}$ is the smooth singular 2-chain defined by

$$\gamma_{g,h}(t,s) = \phi(t(\phi^{-1}(g\phi(s\phi^{-1}(h)))) + s(\phi^{-1}(g\phi((1-t)\phi^{-1}(h)))))$$

This formula defines a smooth function such that $D^2 f(x,y) = \omega(x,y)$. We now only have to check whether f satisfies the local group cocycle identity. Let $(g,h,k) \in \phi(V)^3$ such that gh, hk and ghk are in $\phi(V)$. We have

$$\begin{aligned} f(h,k) - f(gh,k) + f(g,hk) - f(g,h) &= \int_{\gamma_{h,k}} \omega^{inv} - \int_{\gamma_{gh,k}} \omega^{inv} + \int_{\gamma_{g,hk}} \omega^{inv} - \int_{\gamma_{g,h}} \omega^{inv} \\ &= \int_{\partial \gamma_{g,h,k}} \omega^{inv} \end{aligned}$$

where $\gamma_{g,h,k}$ is a smooth singular 3-chain in $\phi(V)$ such that $\partial \gamma_{g,h,k} = g\gamma_{h,k} - \gamma_{gh,k} + \gamma_{g,hk} - \gamma_{g,h}$ (such a chain exists because $\phi(V)$ is homeomorphic to the convex open subset V of $\mathfrak{g}_0$). Thus

$$\begin{aligned} f(h,k) - f(gh,k) + f(g,hk) - f(g,h) &= \int_{\partial \gamma_{g,h,k}} \omega^{inv} \\ &= \int_{\gamma_{g,h,k}} d_{dR} \omega^{inv} \\ &= 0 \end{aligned}$$

because ω^{inv} is a closed 2-form. Hence, we have associated to ω a local group 2-cocycle, smooth in a neighborhood in 1, and such that $D^2 f(x,y) = \omega(x,y)$. Thus we can define a local Lie group structure on $G_0 \times Z(\mathfrak{g})$ by setting

$$(g,a)(h,b) = \big(gh, a + g.b + f(g,h)\big),$$

and the tangent space at $(1,0)$ of this local Lie group is isomorphic to $\mathfrak{g}$. If we want a global structure, we have to extend this local cocycle to the whole group G_0. First P.A. Smith ([Smi50, Smi51]), then W.T. Van Est ([Est62]) have shown that it is precisely this enlargement which may meet an obstruction coming from both $\pi_2(G_0)$ and $\pi_1(G_0)$.

To integrate Leibniz algebras into pointed racks, we follow a similar approach. In this context, we use the fact that we know how to integrate any (finite dimensional) Lie algebra. In a similar way as the Lie algebra case, we associate to any Leibniz algebra an abelian extension of a Lie algebra $\mathfrak{g}_0$ by an anti-symmetric representation $Z_L(\mathfrak{g})$. As we have the theorem for Lie algebras, we can integrate $\mathfrak{g}_0$ and $Z_L(\mathfrak{g})$ into the Lie groups G_0 and $Z_L(\mathfrak{g})$, and, using the Lie's second theorem, $Z_L(\mathfrak{g})$ is a G_0-module. Then, the main difficulty becomes the integration of the Leibniz cocycle into a local Lie rack cocycle. In chapter 3 we explain how to solve this problem. We make a similar construction as in the Lie algebra case, but in this context, there are several difficulties which appear. One of them is that our cocycle is not anti-symmetric, so we can't consider the equivariant form associated to it and integrate this form. To solve this problem, we will use Proposition 1.3.16 which, in particular, establishes an isomorphism from the 2-nd cohomology group of a Leibniz algebra $\mathfrak{g}$ with coefficients in an anti-symmetric representation $\mathfrak{a}^a$ to the 1-st cohomology group of $\mathfrak{g}$ with coefficients in the symmetric representation $Hom(\mathfrak{g}, \mathfrak{a})^s$. In this way, we get a 1-form that we can now integrate. Another difficulty is to specify on which domain this 1-form should be integrated. In the Lie algebra case, we integrate over a 2-simplex and the cocycle identity is verified by integrating over a 3-simplex, whereas in our context we will replace the 2-simplex by the 2-cube and the 3-simplex by a 3-cube.

Chapter 1: Leibniz algebras

This whole chapter, except the last proposition, is based on [Lod93, LP93, Lod98]. We first give the basic definition we need about Leibniz algebra. Unlike J.-L. Loday and T. Pirashvili, who work with right Leibniz algebras, we study the left Leibniz algebras. Hence, we have to translate all the definitions needed in our context. As we have seen above, we translate our integration problem into a cohomological problem, thus we need a cohomology theory for Leibniz algebras and, a fortiori, a notion of representation. We take the definition of a representation over a Leibniz algebra given by J.-L. Loday and T. Pirashvili in [LP93]. In particular, they show in this article the equivalence between the category of representations and the category of modules over an associative algebra denoted $UL(\mathfrak{g})$. Always following [LP93], we define the Leibniz cochain complex of a Leibniz algebra with coefficients in a representation and describe the HL^0, HL^1 and HL^2. In particular, we show that the HL^0 corresponds to the *right invariants*, HL^1 corresponds to the *derivations* modulo the *inner derivations* and the HL^2 corresponds to the *abelian extensions*. Then, we give some canonical abelian extensions associated to a Leibniz algebra (*characteristic extension*, *extension by the left center* and *extension by the center*). We end this chapter by a fondamental result (Proposition 1.3.16). This proposition establishes an isomorphism of cochain complexes from $CL^n(\mathfrak{g}, \mathfrak{a}^a)$ to $CL^{n-1}(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s)$. The important fact in this result is the transfer from an anti-symmetric representation to a symmetric one. This will be useful when we will have to associate a local Lie rack 2-cocycle to a Leibniz 2-cocycle.

Chapter 2: Lie racks

The notion of racks comes from topology, in particular, the theory of invariants of knots and links (cf. for example [FR]). It is M.K. Kinyon in [Kin07] who was the first to link racks to Leibniz algebras. The idea of linking these two structures comes from the Lie groups and Lie algebras case, and particularly, from the construction of the bracket using the conjugation. Indeed, a way to define a bracket on the tangent space at 1 of a Lie group is to differentiate the conjugation morphism twice. Let G a Lie group, the conjugation is the group morphism $c : G \to Aut(G)$ defined by $c_g(h) = ghg^{-1}$. If we differentiate this expression with respect to the variable h at 1, we obtain a Lie group morphism $Ad : G \to Aut(\mathfrak{g})$. We can still derive this morphism at 1 to obtain a linear map $ad : \mathfrak{g} \to End(\mathfrak{g})$. Then, we are allowed to define a bracket $[-,-]$ on $\mathfrak{g}$ by setting $[x, y] = ad(x)(y)$. We can show that this bracket satisfies the left Leibniz identity, and that this identity is induced by the equality $c_g(c_h(k)) = c_{c_g(h)}(c_g(k))$. Thus, if we denote $c_g(h)$ by $g \rhd h$, the only properties we use to define a Lie bracket on $\mathfrak{g}$ are

1. $g\rhd : G \to G$ is a bijection for all $g \in G$.
2. $g\rhd(h\rhd k) = (g\rhd h)\rhd(g\rhd k)$ for all $g, h, k \in G$
3. $g\rhd 1 = 1$ and $1\rhd g = g$ for all $g \in G$.

Hence, we call *(left) rack*, a set provided with a binary operation $\rhd$ satisfying the first and the second condition. A rack is called *pointed* if there exists an element 1 which satisfies the third condition. We begin this chapter by giving definitions and examples, for this we follow [FR]. They work with right racks, hence, as in the Leibniz algebra case, we translate the definitions to left racks. In particular, we give the most important example called *(pointed) augmented rack* (Example 2.1.13). This example presents similarities with crossed modules of groups, and in this case, the rack structure is induced by a group action.

As in the group case, we want to construct a pointed rack associated to a Leibniz algebra using an abelian extension. Hence, we need a cohomology theory where the second cohomology group corresponds to the extension classes of a rack by a module. We take the most general definitions of module and cohomology theory given by N. Jackson in [Jac, Jac07]. These definitions generalize those given first by P. Etingof and M. Graña in [EG03] and secondly by N. Andruskiewitsch and

M. Graña in [AG03]. With these definitions, we translate into the left rack context, the proof given by N. Jackson in [Jac07] which establishes that the second cohomology group classifies the abelian extensions (Theorem 2.3.9). We deduce easily the pointed version of this theorem (Theorem 2.3.17).

A group being a rack, it is natural to ask ourselves whether there exists a link between group cohomology and rack cohomology. The formula (3) reminds us that there exists a morphism between these cohomology theories. In Proposition 2.3.24, we give an explicit formula, which is, to our best knowledge, new, for a morphism of cochain complex from the cochain complex calculating the group cohomology to the cochain complex calculating the rack cohomology. We show in appendix A (section A.7) that this formula gives a morphism of cochain complexes using the trunk theory introduced by R. Fenn, C. Rourke and B. Sanderson in [FRS95].

At the end of this chapter, we give the definitions of local rack cohomology and (local) Lie rack cohomology.

Chapter 3: Lie racks and Leibniz algebras

This chapter is the heart of our thesis. It gives the local solution for the coquecigrue problem. To our knowledge, all the results in this chapter are new, except Proposition 3.1.1 due to M.K. Kinyon ([Kin07]). First, we recall the link between (local) Lie racks and Leibniz algebras explained by M.K. Kinyon in [Kin07] (Proposition 3.1.1). Then, we study the passage from smooth $As(X)$-modules to Leibniz representations (Proposition 3.2.4) and (local) Lie rack cohomology to Leibniz cohomology. We define a morphism from the (local) Lie rack cohomology of a rack X with coefficients in a $As(X)$-module A^s (resp. A^a) to the Leibniz cohomology of the Leibniz algebra associated to X with coefficients in $\mathfrak{a}^s = T_0A$ (resp. $\mathfrak{a}^a$) (Proposition 3.3.1). Then, in the case of symmetric modules, we link this morphism with the morphism $[D^*]$ defined in [Nee04] from group cohomology to Lie algebra cohomology. Precisely, we show in Proposition 3.3.6 that there exists a commutative diagram

$$\begin{tikzcd} H^*_s(G,A) \arrow[r, "{[\Delta^*]}"] \arrow[d, "{[D^*]}"] & HR^*_s(G,A^s) \arrow[d, "{[\delta^n]}"] \\ H^*(\mathfrak{g},\mathfrak{a}) \arrow[r, "{[i^*]}"] & HL^*(\mathfrak{g},\mathfrak{a}^s) \end{tikzcd}$$

where $[i^*]$ is the canonical morphism from Lie algebra cohomology to Leibniz algebra cohomology with coefficients in a symmetric representation. The end of this chapter (section 3.4 to 3.7) is on the integration of Leibniz algebras into local Lie racks. We use the same approach as E. Cartan for the Lie groups case. That is, for every Leibniz algebra, we consider the abelian extension by the left center and integrate it. This extension is caracterised by a 2-cocycle, and we construct (Proposition 3.4.9) a local Lie rack 2-cocycle integrating it by an explicit construction similar to the one explained in the Lie group case. This construction is summarized in our main theorem (Theorem 3.5.3). We remark that the constructed 2-cocycle has more structure (Proposition 3.4.13). That is, the rack cocycle identity is induced by another one. This identity permits us to provide our constructed local Lie rack with a structure of augmented local Lie rack (Proposition 3.6.3). We end this chapter with examples of the integration of non split Leibniz algebras in dimension 4 and 5.

Appendix A: Trunks

The goal of this appendix is to define a morphism of cochain complexes from the cochain complex calculating the group cohomology to the cochain complex calculating the pointed rack cohomology. To reach this goal, we use the trunk theory developped in [FRS95]. This theory permits us to construct by a simplical method (using inclusions of the n-simplex in the n-cube) the wanted morphism.

A trunk is a generalisation of the notion of a category. The idea of trunks is simple: we can see naturally a group G as a category, where the objects and the morphisms are the elements of G, and the composition in the category comes from the product in the group. Because of the axioms of a group, this is a category. The question which comes up naturally is: when we replace the group structure by the rack structure, is it possible to make a similiar construction? Trunk answers this question in the positive.

In the first half of this appendix, we follow [FRS95] to define the trunks, the morphisms between trunks and give some examples. Because they study right racks, they define a trunk theory different from the one we use in this appendix. Therefore our definitions and results are a little bit different from theirs, and are just the translation in the left rack context. The authors introduce an important notion in this theory, namely the nerve of a trunk. This allows them to have a cubical description of (pointed) rack cohomology with coefficients in an $As(X)$-module (Proposition A.6.1).

In the second half, we use this cubical description to construct the morphism from the group cohomology to the pointed rack cohomology of a group defined in Proposition 2.3.24. To do this, we recall a simplicial description of the group cohomology (Proposition A.7.6), and using canonical inclusions of the n-simplex in the n-cube (A.4), we define the wanted morphism (A.5). To our knowledge, this construction is new, and gives a very interesting link between these two cohomology theories.

Chapter 1

Leibniz algebras

1.1 Definitions

Definition 1.1.1. *A* ***Leibniz algebra*** $\mathfrak{g}$ *over* $\mathbb{K}$ *is a* $\mathbb{K}$*-module equipped with a bilinear map, called the bracket,*

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \quad \forall x, y, z \in \mathfrak{g}$$

Remark 1.1.2. There are two definitions for Leibniz algebras, the left Leibniz algebras and the right Leibniz algebras. Here we give the definition of a left Leibniz algebra (we say left because the linear map $[x,-] : \mathfrak{g} \to \mathfrak{g}$ is a derivation for the bracket $[-,-]$). To define a right Leibniz algebra, we ask to the bracket to satisfy the right Leibniz identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$. In this thesis, we consider only left Leibniz algebras over $\mathbb{R}$.

Definition 1.1.3. *Let* $\mathfrak{g}$ *be a Leibniz algebra. A* ***Leibniz subalgebra*** *of* $\mathfrak{g}$ *is a vector subspace* $\mathfrak{h}$ *of* $\mathfrak{g}$ *such that* $[x,y] \in \mathfrak{h} \quad \forall x, y \in \mathfrak{h}$.
A ***left*** *(resp.* ***right****)* ***ideal*** *of* $\mathfrak{g}$ *is a vector subspace* $\mathfrak{h}$ *such that* $[x,y]$ *(resp.* $[y,x]$*)* $\in \mathfrak{h} \quad \forall x \in \mathfrak{h}, y \in \mathfrak{g}$.
A ***ideal*** *in* $\mathfrak{g}$ *is a left and right ideal in* $\mathfrak{g}$.

Proposition 1.1.4. *Let* $\mathfrak{g}$ *be a Leibniz algebra, we have* $[[x,x],y] = 0 \quad \forall x, y \in \mathfrak{g}$.

Proof : This equality comes from the Leibniz identity. For $x, y \in \mathfrak{g}$, we have

$$[x,[x,y]] = [[x,x],y] + [x,[x,y]]$$

Thus $[[x,x],y] = 0$.

$\square$

Let $\mathfrak{g}$ be a Leibniz algebra. If we suppose that the bracket is anti-symmetric, then the Leibniz identity is equivalent the Jacobi identity. Hence, a Lie algebra is a Leibniz algebra. On other hand, the obstruction for a Leibniz algebra to be a Lie algebra is measured by an ideal denoted $\mathfrak{g}_{ann}$. By definition, $\mathfrak{g}_{ann}$ is the right ideal generated by the set $\{[x,x] \in \mathfrak{g} \,|\, x \in \mathfrak{g}\}$. Because of Proposition 1.1.4, $\mathfrak{g}_{ann}$ is a left ideal, thus $\mathfrak{g}_{ann}$ is an ideal of $\mathfrak{g}$, and the quotient of $\mathfrak{g}$ by $\mathfrak{g}_{ann}$ is a Lie algebra denoted by $\mathfrak{g}_{Lie}$. We remark that if $\mathfrak{g}$ is a Lie algebra, then $\mathfrak{g}_{ann}$ is zero and $\mathfrak{g}$ is equal to $\mathfrak{g}_{Lie}$.

Definition 1.1.5. *Let $\mathfrak{g}$ and $\mathfrak{h}$ two Leibniz algebras. A **morphism** of Leibniz algebras $f : \mathfrak{g} \to \mathfrak{h}$ is a linear map which respects the bracket, that is*

$$f([x,y]) = [f(x), f(y)] \quad \forall x, y \in \mathfrak{g}$$

Example 1.1.6. If $\mathfrak{g}$ is a Leibniz algebra, then $ad : \mathfrak{g} \to End(\mathfrak{g})$ is a morphism of Leibniz algebra. Indeed, we have $\forall x, y, z \in \mathfrak{g}, ad([x,y])(z) = [[x,y],z] = [x,[y,z]] - [y,[x,z]] = (ad(x) \circ ad(y) - ad(y) \circ ad(x))(z)$. Hence, $ad([x,y]) = [ad(x), ad(y)] \quad \forall x, y \in \mathfrak{g}$, thus ad is a Leibniz algebra morphism.

Proposition 1.1.7. *Let $f : \mathfrak{g} \to \mathfrak{h}$ be a Leibniz algebra morphism, then $Ker(f)$ is an ideal in $\mathfrak{g}$.*

Proof : Let $x \in Ker(f), y \in \mathfrak{g}$, we have $f([x,y]) = f([y,x]) = [f(x), f(y)] = 0$. Hence, $Ker(f)$ is an ideal in $\mathfrak{g}$.

□

1.2 Representations

1.2.1 Definition

Definition 1.2.1. *Let $\mathfrak{g}$ be a Leibniz algebra, a $\mathfrak{g}$-representation is a vector space M equipped with two bilinear maps*

$$[-,-]_L : \mathfrak{g} \times M \to M \text{ and } [-,-]_R : M \times \mathfrak{g} \to M$$

satisfying the following three axioms:

$$[x,[y,m]_L]_L = [[x,y],m]_L + [y,[x,m]_L]_L \qquad (LLM)$$
$$[x,[m,y]_R]_L = [[x,m]_L,y]_R + [m,[x,y]]_R \qquad (LML)$$
$$[m,[x,y]]_R = [[m,x]_R,y]_R + [x,[m,y]_R]_L \qquad (MLL)$$

Remark 1.2.2. The axioms (LML) and (MLL) implies the relation

$$[[x,m]_L,y]_R + [[m,x]_R,y]_R = 0 \qquad (ZD)$$

Example 1.2.3. Let $(\mathfrak{g}, [-,-])$ be a Leibniz algebra, then $\mathfrak{g}$ is a $\mathfrak{g}$-representation where $[-,-]_L = [-,-]_R = [-,-]$.

Definition 1.2.4. *Let $\mathfrak{g}$ be a Leibniz algebra and let M be a $\mathfrak{g}$-representation. M is called **symmetric** when*

$$[x,m]_L = -[m,x]_R \quad \forall x \in \mathfrak{g}, m \in M$$

*M is called **anti-symmetric** when*

$$[m,x]_R = 0 \quad \forall x \in \mathfrak{g}, m \in M$$

*M is called **trivial** when it is symmetric and anti-symmetric. That is,*

$$[x,m]_L = [m,x]_R = 0$$

Remark 1.2.5. Let $\mathfrak{g}$ be a Lie algebra and let M be a $\mathfrak{g}$-module (in the Lie sense). Then we can consider M as a symmetric $\mathfrak{g}$-representation putting

$$[x,m]_L = -[m,x]_R = [x,m] \quad \forall x \in \mathfrak{g}, m \in M$$

and we can consider M as an anti-symmetric $\mathfrak{g}$-representation putting

$$[x,m]_L = [x,m] \text{ and } [m,x]_R = 0 \quad \forall x \in \mathfrak{g}, m \in M$$

1.2.2 Universal enveloping algebra of a Leibniz algebra

Let $\mathfrak{g}$ be a Lie algebra, a $\mathfrak{g}$-module can be viewed as a module over the associative and unital algebra $U(\mathfrak{g})$. In the Leibniz algebras case, we have the same kind of result.

Definition 1.2.6. *Let $\mathfrak{g}$ be a Leibniz algebra. The universal enveloping algebra of $\mathfrak{g}$ is the associative and unital algebra*

$$UL(\mathfrak{g}) := T(\mathfrak{g} \oplus \mathfrak{g})/I$$

where $T(\mathfrak{g} \oplus \mathfrak{g})$ is the tensor algebra $\oplus_n(\mathfrak{g} \oplus \mathfrak{g})^{\otimes^n}$ and I is the two-sided ideal generated by the relations

$$0 = ([x, y], 0) - (x, 0) \otimes (y, 0) + (y, 0) \otimes (x, 0) \tag{1.1}$$

$$0 = (0, [x, y]) - (x, 0) \otimes (0, y) + (0, y) \otimes (x, 0) \tag{1.2}$$

$$0 = (0, y) \otimes (x, x) \tag{1.3}$$

Theorem 1.2.7. *Let $\mathfrak{g}$ be a Leibniz algebra. The category of $\mathfrak{g}$-representations is isomorphic to the category of left $UL(\mathfrak{g})$-modules.*

Proof : Let M be a $\mathfrak{g}$-representation. We have to define a morphism of unital and associative algebras

$$UL(\mathfrak{g}) \to End(M)$$

We define a linear map $\mathfrak{g} \oplus \mathfrak{g} \to End(M)$ putting

$$(x, y) \mapsto (m \mapsto [x, m]_L + [m, y]_R)$$

This map extended in a unique way to a morphism of algebra

$$T(\mathfrak{g} \oplus \mathfrak{g}) \to End(M)$$

By the axiom (LLM) (resp (LML)), (1.1) (resp. (1.2)) is sent to zero. Moreover, in the presence of (LML), the axiom (MLL) is equivalent to (ZD). Thus (1.3) is sent to zero, too. Hence, there is a morphism

$$UL(\mathfrak{g}) \to End(M)$$

Conversely, if we have a left $UL(\mathfrak{g})$-module, we define two linear maps $[-, -]_L$ and $[-, -]_R$ putting

$$[x, m]_L = (x, 0).m \text{ and } [m, x]_R = (0, x).m \quad \forall x \in \mathfrak{g}, m \in M$$

The fact that these two linear maps satisfiy the axioms $(LLM), (LML)$ and (MLL) is easily verified.

□

In the first section, we have seen that there is a Lie algebra $\mathfrak{g}_{Lie}$ associated to $\mathfrak{g}$ and we can consider the universal enveloping algebra $U(\mathfrak{g}_{Lie})$. The following proposition establishes morphisms between $UL(\mathfrak{g})$ and $U(\mathfrak{g}_{Lie})$.

Proposition 1.2.8. *Let $\mathfrak{g}$ be a Leibniz algebra. There are algebra homomorphisms*

$$d_0, d_1 : UL(\mathfrak{g}) \to U(\mathfrak{g}_{Lie}) \text{ and } s_0 : U(\mathfrak{g}_{Lie}) \to UL(\mathfrak{g})$$

which satisfy

$$d_0 s_0 = d_1 s_0 = id$$

Proof : Define $d_0, d_1 : UL(\mathfrak{g}) \to U(\mathfrak{g}_{Lie})$ and $s_0 : U(\mathfrak{g}_{Lie}) \to UL(\mathfrak{g})$ by

$$d_0(x,0) = \bar{x},\ d_0(0,x) = 0$$

$$d_1(x,0) = \bar{x},\ d_0(0,x) = -\bar{x}$$

and

$$s_0(\bar{x}) = (x,0)$$

It is clear that d_0, d_1 and s_0 are well-defined algebra homomorphisms (since $([x,x],0) = (0,0)$).

□

Remark 1.2.9. Let M be a $\mathfrak{g}_{Lie}$-module (in the Lie sense), then this proposition gives us two ways to define on M a structure of $\mathfrak{g}$-representation. Indeed, a $\mathfrak{g}_{Lie}$-module M is a vector space with a morphism of algebras $U(\mathfrak{g}_{Lie}) \xrightarrow{\rho} End(M)$. To have a structure of $\mathfrak{g}$-representation on M, we have to define a morphism of algebras $UL(\mathfrak{g}) \xrightarrow{\lambda} End(M)$. In our case, to define λ, we have two canonical choices. One is to compose ρ with d_0 and the other is to compose ρ with d_1. The first gives to M a structure of anti-symmetric $\mathfrak{g}$-representation and the second a structure of symmetric $\mathfrak{g}$-representation. In the case where $\mathfrak{g}$ is a Lie algebra we find the result in Remark 1.2.5.

1.3 Cohomology of Leibniz algebras

1.3.1 The cochain complex

Let $\mathfrak{g}$ be a Leibniz algebra and let M be a $\mathfrak{g}$-representation. We define a cochain complex $\{CL^n(\mathfrak{g},M), d_L^n\}_{n\geq 0}$ putting

$$CL^n(\mathfrak{g},M) := Hom(\mathfrak{g}^{\otimes^n}, M)$$

and

$$d_L^n : CL^n(\mathfrak{g},M) \to CL^{n+1}(\mathfrak{g},M)$$

where

$$\begin{aligned} d_L^n\omega(x_0,\ldots,x_n) &= \sum_{i=0}^{n-1}(-1)^i[x_i,\omega(x_0,\ldots,\hat{x_i},\ldots,x_n)]_L + (-1)^{n-1}[\omega(x_0,\ldots,x_{n-1}),x_n]_R \\ &+ \sum_{0\leq i<j\leq n}(-1)^{i+1}\omega(x_0,\ldots,x_{j-1},[x_i,x_j],x_{j+1},\ldots,x_n) \end{aligned}$$

Lemma 1.3.1. $d_L^{n+1} \circ d_L^n = 0$

To prove this lemma, we use *Cartan's formulas*. For all $y \in \mathfrak{g}$ and $n \in \mathbb{N}$, we define two linear maps $\theta(y) : CL^n(\mathfrak{g},M) \to CL^n(\mathfrak{g},M)$ and $i(y) : CL^{n+1}(\mathfrak{g},M) \to CL^n(\mathfrak{g},M)$ by

$$\theta(y)(f)(x_1,\ldots,x_n) = [y, f(x_1,\ldots,x_n)]_L - \sum_{i=1}^{n} f(x_1,\ldots,[y,x_i],\ldots,x_n)$$

and

$$i(y)(f)(x_1,\ldots,x_n) = f(y,x_1,\ldots,x_n)$$

Proposition 1.3.2 (Cartan's formulas)**.** *We have the following identities*

1. $d_L^{n-1} \circ i(y) + i(y) \circ d_L^n = \theta(y)$.
2. $\theta(x) \circ \theta(y) - \theta(y) \circ \theta(x) = \theta([x, y])$ *for* $n > 0$.
3. $\theta(x) \circ i(y) - i(y) \circ \theta(x) = i([x, y])$ *for* $n > 0$.
4. $\theta(y) \circ d_L^n = d_L^n \circ \theta(y)$.

Proof of Cartan's formulas:

1. For $n = 1$ we have $d_L^0(i(y)(f))(x_1) = -[f(y), x_1]_R$ and $i(y)(d_L^1(f))(x_1) = d_L^1(f)(y, x_1) = [y, f(x_1)]_L + [f(y), x_1]_R - f([y, x_1])$. Hence, $d_L^0 \circ i(y) + i(y) \circ d_L^1 = \theta(y)$.
   Now let $n > 1$, we have

$$\begin{aligned} d_L^{n-1}(i(y)(f))(x_1, \ldots, x_n) &= \sum_{i=1}^{n-1} (-1)^i [x_i, i(y)(f)(x_1, \ldots, \widehat{x_i}, \ldots, x_n)]_L \\ &+ (-1)^{n-1}[i(y)(f)(x_1, \ldots, x_{n-1}), x_n]_R \\ &+ \sum_{1 \le i < j \le n} (-1)^i i(y)(f)(x_1, \ldots, [x_i, x_j], \ldots, x_n) \\ &= \sum_{i=1}^{n-1} (-1)^i [x_i, f(y, x_1, \ldots, \widehat{x_i}, \ldots, x_n)]_L \\ &+ (-1)^{n-1}[f(y, x_1, \ldots, x_{n-1}), x_n]_R \\ &+ \sum_{1 \le i < j \le n} (-1)^i f(y, x_1, \ldots, [x_i, x_j], \ldots, x_n) \end{aligned}$$

   and

$$\begin{aligned} i(y)(d_L^n(f))(x_1, \ldots, x_n) &= [y, f(x_1, \ldots, x_n)]_L + \sum_{i=1}^{n-1} (-1)^{i-1} [x_i, f(y, x_1, \ldots, \widehat{x_i}, \ldots, x_n)]_L \\ &+ (-1)^n [f(y, x_1, \ldots, x_{n-1}), x_n]_R - \sum_{1 \le i \le n} f(x_1, \ldots, [y, x_i], \ldots, x_n) \\ &+ \sum_{1 \le i < j \le n} (-1)^{i-1} f(y, x_1, \ldots, [x_i, x_j], \ldots, x_n) \end{aligned}$$

   Thus $d_L^{n-1} \circ i(y) + i(y) \circ d_L^n = \theta(y)$.

2. We have

$$\theta(x)(\theta(y)(f))(x_1, \ldots, x_n) = [x, \theta(y)(f)(x_1, \ldots, x_n)]_L - \sum_{i=1}^{n} \theta(y)(f)(x_1, \ldots, [x, x_i], \ldots, x_n)$$

$$\begin{aligned}\theta(x)(\theta(y)(f))(x_1,\ldots,x_n) = {} & [x,[y,f(x_1,\ldots,x_n)]_L]_L - \sum_{i=1}^{n}[x,f(x_1,\ldots,[y,x_i],\ldots,x_n)]_L \\ & - \sum_{i=1}^{n}[y,f(x_1,\ldots,[x,x_i],\ldots,x_n)]_L \\ & + \sum_{i,j=1,j\neq i}^{n} f(x_1,\ldots,[y,x_j],\ldots,[x,x_i],\ldots,x_n) \\ & + \sum_{i=1}^{n} f(x_1,\ldots,[y,[x,x_i]],\ldots,x_n)\end{aligned}$$

Using the Leibniz identity and the axiom (LML), we obtain

$$\begin{aligned}(\theta(x)(\theta(y)(f)) - \theta(y)(\theta(x)(f)))(x_1,\ldots,x_n) &= [[x,y],f(x_1,\ldots,x_n)]_L - \sum_{i=1}^{n} f(x_1,\ldots,[[x,y],x_i],\ldots,x_n) \\ &= \theta([x,y])(f)(x_1,\ldots,x_n)\end{aligned}$$

3. We have

$$\begin{aligned}\theta(x)(i(y)(f))(x_1,\ldots,x_n) &= [x,i(y)(f)(x_1,\ldots,x_n)]_L - \sum_{i=1}^{n} i(y)(f)(x_1,\ldots,[x,x_i],\ldots,x_n) \\ &= [x,f(y,x_1,\ldots,x_n)]_L - \sum_{i=1}^{n} f(y,x_1,\ldots,[x,x_i],\ldots,x_n)\end{aligned}$$

and

$$\begin{aligned}i(y)(\theta(x)(f))(x_1,\ldots,x_n) &= \theta(x)(f)(y,x_1,\ldots,x_n) \\ &= [x,f(y,x_1,\ldots,x_n)]_L - f([x,y],x_1,\ldots,x_n) \\ &\quad - \sum_{i=1}^{n} f(y,x_1,\ldots,[x,x_i],\ldots,x_n)\end{aligned}$$

Thus $(\theta(x)\circ i(y) - i(y)\circ\theta(x))(f)(x_1,\ldots,x_n) = f([x,y],x_1,\ldots,x_n)$. That is $\theta(x)\circ i(y) - i(y)\circ\theta(x) = i([x,y])$.

4. We proceed by induction on n. For $n=0$, we have

$$\begin{aligned}\theta(y)(d_L^0(m))(x_1) &= [y,d_L^0(m)(x_1)]_L - d_L^0(m)([y,x_1]) \\ &= -[y,[m,x_1]_R]_L + [m,[y,x_1]]_R\end{aligned}$$

and

$$\begin{aligned}d_L^0(\theta(y)(m))(x_1) &= [\theta(y)(m),x_1]_R \\ &= [[m,y]_R,x_1]_R\end{aligned}$$

and using the axiom (MLL) we have $\theta(y)\circ d_L^0 = d_L^0\circ\theta(y)$. Now suppose that $n>0$, we have

$$(d_L^n\circ\theta(y) - \theta(y)\circ d_L^n)(f)(x_1,\ldots,x_n) = (i(x_1)\circ d_L^n\circ\theta(y) - i(x_1)\circ\theta(y)\circ d_L^n)(f)(x_2,\ldots,x_n)$$

Hence it is sufficient to show that

$$i(x) \circ d_L^n \circ \theta(y) - i(x) \circ \theta(y) \circ d_L^n = 0$$

We have

$$\begin{aligned} i(x) \circ d_L^n \circ \theta(y) - i(x) \circ \theta(y) \circ d_L^n &= \theta(x) \circ \theta(y) - d_L^{n-1} \circ i(x) \circ \theta(y) + i([y,x]) \circ d_L^n \\ &- \theta(y) \circ i(x) \circ d_L^n \\ &= \theta(x) \circ \theta(y) - d_L^{n-1} \circ i(x) \circ \theta(y) + \theta([y,x]) - d_L^{n-1} \circ i([y,x]) \\ &+ \theta(y) \circ d_L^{n-1} \circ i(x) - \theta(y) \circ \theta(x) \\ &= -d_L^{n-1} \circ i(x) \circ \theta(y) - d_L^{n-1} \circ i([y,x]) + \theta(y) \circ d_L^{n-1} \circ i(x) \\ &= -d_L^{n-1} \circ i(x) \circ \theta(y) - d_L^{n-1} \circ i([y,x]) + d_L^{n-1} \circ \theta(y) \circ i(x) \\ &= 0 \end{aligned}$$

Hence $\theta(y) \circ d_L^n = d_L^n \circ \theta(y)$.

□

Proof of the Lemma: We have in low dimension

$$\begin{aligned} d_L^1(d_L^0 m)(x_1, x_2) &= [x_1, d_L^0 m(x_2)]_L + [d_L^0 m(x_1), x_2]_R - d_L^0 f([x_1, x_2]) \\ &= -[x_1, [m, x_2]_R]_L - [[m, x_1]_R, x_2]_R + [m, [x_1, x_2]]_R \\ &= 0 \text{ by axiom } (MLL) \end{aligned}$$

To prove the general case, we proceed by induction. We have

$$d_L^{n+1}(d_L^n(f))(x_1, \ldots, x_{n+1}) = (i(x_1) \circ d_L^{n+1} \circ d_L^n)(f)(x_2, \ldots, x_{n+1})$$

but by (1) we obtain

$$\begin{aligned} i(x_1) \circ d_L^{n+1} \circ d_L^n &= \theta(x_1) \circ d_L^n - d_L^n \circ i(x_1) \circ d_L^n \\ &= \theta(x_1) \circ d_L^n - d_L^n \circ \theta(x_1) + d_L^n \circ d_L^n \circ i(x_1) \\ &= 0 \end{aligned}$$

□

Definition 1.3.3. *Let $\mathfrak{g}$ be a Leibniz algebra and let M be a $\mathfrak{g}$-representation. The cohomology of $\mathfrak{g}$ with coefficients in M is the cohomology of the cochain complex $\{CL^n(\mathfrak{g}, M), d_L^n\}_{n \geq 0}$.*

$$HL^n(\mathfrak{g}, M) := H^n(\{CL^n(\mathfrak{g}, M), d_L^n\}_{n \geq 0}) \quad \forall n \geq 0$$

1.3.2 A morphism from Lie cohomology to Leibniz cohomology

A Lie algebra $\mathfrak{g}$ being a Leibniz algebra, it is natural to hope a link between Lie cohomology and Leibniz cohomology of $\mathfrak{g}$. Recall (cf. [HS97, CE56]) that the Lie cohomology of Lie algebra is defined as the cohomology of the cochain complex $\{C^n(\mathfrak{g}, M), d^n\}_{n \geq 0}$ where

$$C^n(\mathfrak{g}, M) = Hom(\Lambda^n \mathfrak{g}, M)$$

and

$$
\begin{aligned}
d^n\omega(x_0,\ldots,x_n) &= \sum_{i=0}^{n}(-1)^i[x_i,\omega(x_0,\ldots,\hat{x_i},\ldots,x_n)] \\
&+ \sum_{0\le i<j\le n}(-1)^{i+1}\omega(x_0,\ldots,x_{j-1},[x_i,x_j],x_{j+1},\ldots,x_n)
\end{aligned}
$$

In Remark 1.2.5, we have seen that we can give to a Lie module M two stuctures of Leibniz module. One is symmetric and the other is antisymmetric. The following proposition establishes a morphism from the Lie cohomology of a Lie algebra $\mathfrak{g}$ with coefficients in a Lie module M to the Leibniz cohomolgy of $\mathfrak{g}$ with coefficients in the Leibniz representation M^s.

Proposition 1.3.4. *Let $\mathfrak{g}$ be a Lie algebra and let M be a $\mathfrak{g}$-module. We have a morphism of cochain complexes*

$$C^n(\mathfrak{g},M) \xrightarrow{i^n} CL^n(\mathfrak{g},M^s)$$

given by the canonical inclusion of $Hom(\Lambda^n\mathfrak{g},M)$ into $Hom(\mathfrak{g}^{\otimes n},M)$.

Proof : The formula for the differentials is the same. Hence, the result is clear.

□

1.3.3 HL^0, HL^1 and HL^2

HL^0

Let $\mathfrak{g}$ be a Leibniz algebra and let M be a $\mathfrak{g}$-representation. By definition, $CL^0(\mathfrak{g},M)=M$ and $d^0_L m(x) = -[m,x]_R$. Hence,

$$HL^0(\mathfrak{g},M) = ZL^0(\mathfrak{g},M) = \{m\in M | [m,x]_R = 0 \ \ \forall x\in\mathfrak{g}\}$$

It is called the submodule of the *right invariants*. We remark that if M is anti-symmetric, then $HL^0(\mathfrak{g},M)=M$.

HL^1

Let $\mathfrak{g}$ be a Leibniz algebra and let M be a $\mathfrak{g}$-representation. By definition,

$$d^1_L(\omega)(x,y) = [x,\omega(y)]_L + [\omega(x),y]_R - \omega([x,y])$$

Hence,

$$ZL^1(\mathfrak{g},M) = \{\omega\in Hom(\mathfrak{g},M) | [x,\omega(y)]_L + [\omega(x),y]_R = \omega([x,y]) \ \ \forall x,y\in\mathfrak{g}\}$$

An element of $ZL^1(\mathfrak{g},M)$ is called a *derivation* from $\mathfrak{g}$ to M, and the module of derivations is denoted by $Der(\mathfrak{g},M)$. Moreover, we have

$$BL^1(\mathfrak{g},M) = \{\omega\in Hom(\mathfrak{g},M) | \omega(x) = [m,x]_R \, \forall x\in\mathfrak{g}\}$$

An element in $BL^1(\mathfrak{g},M)$ is called an *inner derivation* and the module of inner derivations is denoted by $InnDer(\mathfrak{g},M)$. Finally, we have

$$HL^1(\mathfrak{g},M) = Der(\mathfrak{g},M)/InnDer(\mathfrak{g},M)$$

HL^2

Definition 1.3.5. *An abelian extension of Leibniz algebras is a short exact sequence of Leibniz algebras* $0 \to M \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$ *such that* M *is an abelian Leibniz algebra* ($[M, M] = 0$).

Proposition 1.3.6. *Let* $0 \to M \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ *be an abelian extension of Leibniz algebras, then* M *is a* $\mathfrak{g}$*-representation.*

Proof : We have to define two bilinear maps $[-,-]_L : \mathfrak{g} \times M \to M$ and $[-,-]_R : M \times \mathfrak{g} \to M$ which satisfy the axioms $(LLM), (LML)$ and (MLL). We put

$$[x, m]_L = i^{-1}([s(x), i(m)]_{\hat{\mathfrak{g}}}) \text{ and } [m, x]_R = i^{-1}([i(m), s(x)]_{\hat{\mathfrak{g}}})$$

where s is a linear section of p.
The first thing we have to show is that these actions don't depend on the section.
Let s' another section of p, we have $s(x) - s'(x) \in i(M)$, so $[s(x) - s'(x), i(m)]_{\hat{\mathfrak{g}}} = 0$ and $[i(m), s(x) - s'(x)]_{\hat{\mathfrak{g}}} = 0$.
For the axioms (LLM), we have to show that $[[x, y]_{\mathfrak{g}}, m]_L = [x, [y, m]_L]_L - [y, [x, m]_L]_L$. We have

$$i([[x, y]_{\mathfrak{g}}, m]_L) = [s([x, y]_{\mathfrak{g}}), i(m)]_{\hat{\mathfrak{g}}}$$

and

$$i([x, [y, m]_L]_L - [y, [x, m]_L]_L) = [s(x), [s(y), i(m)]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}} - [s(y), [s(x), i(m)]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}} = [[s(x), s(y)]_{\hat{\mathfrak{g}}}, i(m)]_{\hat{\mathfrak{g}}}$$

But $s([x, y]_{\mathfrak{g}}) - [s(x), s(y)]_{\hat{\mathfrak{g}}} \in i(M)$, so $[s([x, y]_{\mathfrak{g}}), i(m)]_{\hat{\mathfrak{g}}} = [[s(x), s(y)]_{\hat{\mathfrak{g}}}, i(m)]_{\hat{\mathfrak{g}}}$. Using the Leibniz identity, we have (LLM).
For the axioms (LML) and (MLL), the argument is similar.

□

Definition 1.3.7. *Let* $\mathfrak{g}$ *be a Leibniz algebra and let* M *be a* $\mathfrak{g}$*-representation. An* ***abelian extension of*** $\mathfrak{g}$ ***by*** M *is a short exact sequence of Leibniz algebras*

$$0 \to M \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$$

such that M *is abelian and the action of* $\mathfrak{g}$ *on* M *induced by the extension coincides.*

Definition 1.3.8. *Let* $\mathfrak{g}$ *be a Leibniz algebra and let* M *be a* $\mathfrak{g}$*-representation. We say that two abelian extensions of* $\mathfrak{g}$ *by* M, $0 \to M \xrightarrow{i_1} \hat{\mathfrak{g}}_1 \xrightarrow{p_1} \mathfrak{g} \to 0$ *and* $0 \to M \xrightarrow{i_2} \hat{\mathfrak{g}}_2 \xrightarrow{p_2} \mathfrak{g} \to 0$ *are* ***equivalent*** *if there exists a morphism* $\hat{\mathfrak{g}}_1 \xrightarrow{\phi} \hat{\mathfrak{g}}_2$ *such that the following diagram commutes*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_1"] \arrow[d, "id"'] & \hat{\mathfrak{g}}_1 \arrow[r, "p_1"] \arrow[d, "\phi"] & \mathfrak{g} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & M \arrow[r, "i_2"] & \hat{\mathfrak{g}}_2 \arrow[r, "p_2"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Remark 1.3.9. The morphism ϕ is necessarily a isomorphism. Indeed, let $x \in \hat{\mathfrak{g}}_1$ such that $\phi(x) = 0$. We have $p_2(\phi(x)) = p_1(x) = 0$, so there exists $m \in M$ such that $x = i_1(m)$, and, because of $\phi(i_1(m)) = i_2(m) = 0$ and the injectivity of i_2, we have $x = 0$. Now, let $y \in \hat{\mathfrak{g}}_2$, we have $y = \sigma_2(p_2(y)) + y - \sigma_2(p_2(y))$ where $\sigma_2 = \phi \circ \sigma_1$ with σ_1 a linear section of p_1. Let $x = \sigma_1(p_2(y))$ and $m \in M$ such that $i_2(m) = y - \sigma_2(p_2(y))$, we have $p_2(\phi(x + i_1(m))) = p_2(y)$ and $\phi(x+i_1(m)) - \sigma_2(p_2(\phi(x+i_1(m)))) = y - \sigma_2(p_2(y))$. So $\phi(x+i_1(m)) = y$ and ϕ is surjective.

We denote by $Ext(\mathfrak{g}, M)$ the set of equivalence classes of extensions of $\mathfrak{g}$ by M.

Example 1.3.10. Let $\mathfrak{g}$ be a Leibniz algebra, let M be a $\mathfrak{g}$-representation and $\omega \in ZL^2(\mathfrak{g}, M)$, then we can define an abelian extension of $\mathfrak{g}$ by M

$$0 \to M \xrightarrow{i} \mathfrak{g} \oplus_\omega M \xrightarrow{p} \mathfrak{g} \to 0$$

where the Leibniz bracket on $\mathfrak{g} \oplus_\omega M$ is defined by

$$[(x, m), (x', m')] = ([x, x'], [x, m']_L + [m, x']_R + \omega(x, x'))$$

This bracket satisfies the Leibniz identity because of the axioms $(LLM), (LML), (MLL)$ and the cocycle identity. Indeed, we have

$$\begin{aligned}
[(x, m), [(x', m'), (x'', m'')]] &= [(x, m), ([x', x''], [x', m'']_L + [m', x'']_R + \omega(x', x''))] \\
&= ([x, [x', x'']], [x, [x', m'']_L]_L + [x, [m', x'']_R]_L \\
&+ [x, \omega(x', x'')]_L + [m, [x', x'']]_R + \omega([x, [x', x'']]))
\end{aligned}$$

$$\begin{aligned}
[[(x, m), (x', m')], (x'', m'')] &= [([x, x'], [x, m']_L + [m, x']_R + \omega(x, x')), (x'', m'')] \\
&= ([[x, x'], x''], [[x, x'], m'']_L + [[x, m']_L, x'']_R \\
&+ [[m, x']_R, x'']_R + [\omega(x, x'), x'']_R + \omega([x, x'], x'')
\end{aligned}$$

$$\begin{aligned}
[(x', m'), [(x, m), (x'', m'')]] &= [(x', m'), ([x, x''], [x, m'']_L + [m, x'']_R + \omega(x, x''))] \\
&= ([x', [x, x'']], [x', [x, m'']_L]_L + [x', [m, x'']_R]_L \\
&+ [x', \omega(x, x'')]_L + [m', [x, x'']]_R + \omega([x', [x, x'']]))
\end{aligned}$$

Now

$$\begin{aligned}
(LLM) \Rightarrow \quad & [x, [x', m'']_L]_L = [[x, x'], m'']_L + [x', [x, m'']_L]_L \\
(LML) \Rightarrow \quad & [x, [m', x'']_R]_L = [[x, m']_L, x'']_R + [m', [x, x'']]_R \\
(MLL) \Rightarrow \quad & [m, [x', x'']]_R = [[m, x']_R, x'']_R + [x', [m, x'']_R]_L
\end{aligned}$$

and

$$\begin{aligned}
\omega \in ZL^2(\mathfrak{g}, M) \Rightarrow [x, \omega(x', x'')]_L + \omega([x, [x', x'']]) &= [\omega(x, x'), x'']_R + \omega([x, x'], x'') \\
&+ [x', \omega(x, x'')]_L + \omega([x', [x, x'']])
\end{aligned}$$

Hence the Leibniz identity is satisfied.

Proposition 1.3.11. *Let $\mathfrak{g}$ be a Leibniz algebra and let M be a $\mathfrak{g}$-module. Every class of abelian extensions in $Ext(\mathfrak{g}, M)$ can be represented by an abelian extension of the form $0 \to M \xrightarrow{i} \mathfrak{g} \oplus_\omega M \xrightarrow{p} \mathfrak{g} \to 0$.*

Proof : Let $0 \to M \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ an abelian extension of $\mathfrak{g}$ by M and σ a linear section of this short exact sequence. We define a linear map $\omega : \mathfrak{g} \otimes \mathfrak{g} \to M$ putting

$$\omega(x, x') = i^{-1}([\sigma(x), \sigma(x')] - \sigma([x, x']))$$

We have

$$\begin{aligned}
d_L\omega(x,x',x'') &= [x,\omega(x',x'')]_L + \omega([x,[x',x'']]) - [\omega(x,x'),x'']_R \\
&- \omega([x,x'],x'') - [x',\omega(x,x'')]_L - \omega([x',[x,x'']]) \\
&= i^{-1}([\sigma(x), i(\omega(x',x''))]) + \omega([x,[x',x'']]) - i^{-1}([i(\omega(x,x')),\sigma(x'')]) \\
&- \omega([x,x'],x'') - i^{-1}([\sigma(x'), i(\omega(x,x''))]) - \omega([x',[x,x'']]) \\
&= i^{-1}([\sigma(x),[\sigma(x'),\sigma(x'')] - \sigma([x',x''])]) + i^{-1}([\sigma(x),\sigma([x',x''])] - \sigma([x,[x',x'']])) \\
&- i^{-1}([[\sigma(x),\sigma(x')] + \sigma([x,x']),\sigma(x'')]) - i^{-1}([\sigma([x,x']),\sigma(x'')] + \sigma([[x,x'],x''])) \\
&- i^{-1}([\sigma(x'),[\sigma(x),\sigma(x'')] + \sigma([x,x''])]) - i^{-1}([\sigma(x'),\sigma([x,x''])] + \sigma([x',[x,x'']])) \\
&= i^{-1}([\sigma(x),[\sigma(x'),\sigma(x'')]] - [\sigma(x),\sigma([x',x''])] + [\sigma(x),\sigma([x',x''])] - \sigma([x,[x',x'']]) \\
&- [[\sigma(x),\sigma(x')],\sigma(x'')] + [\sigma([x,x']),\sigma(x'')] - [\sigma([x,x']),\sigma(x'')] + \sigma([[x,x'],x'']) \\
&- [\sigma(x'),[\sigma(x),\sigma(x'')]] + [\sigma(x'),\sigma([x,x''])] - [\sigma(x'),\sigma([x,x''])] + \sigma([x',[x,x'']])) \\
&= i^{-1}(0) \\
&= 0
\end{aligned}$$

Thus $\omega \in ZL^2(\mathfrak{g},M)$, and we can consider the abelian extension $0 \to M \xrightarrow{i} \mathfrak{g} \oplus_\omega M \xrightarrow{p} \mathfrak{g} \to 0$. Now, we want to show that this extension is equivalent to $0 \to M \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$, that is, we have to find a morphism $\mathfrak{g} \oplus_\omega M \xrightarrow{\phi} \hat{\mathfrak{g}}$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "id"'] & \mathfrak{g} \oplus_\omega M \arrow[r] \arrow[d, "\phi"] & \mathfrak{g} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] & \hat{\mathfrak{g}} \arrow[r, "p"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

We take

$$\phi(x,m) = i(m) + \sigma(x)$$

Clearly this diagram commutes. Moreover, we have

$$\begin{aligned}
\phi([(x,m),(x',m')]) &= \phi([x,x'],[x,m'] + [m,x'] + \omega(x,x')) \\
&= i([x,m']) + i([m,x']) + i(\omega(x,x')) + \sigma([x,x']) \\
&= [\sigma(x), i(m')] + [i(m),\sigma(x')] + [\sigma(x),\sigma(x')] - \sigma([x,x']) + \sigma([x,x']) \\
&= [i(m) + \sigma(x), i(m') + \sigma(x')] \\
&= [\phi(x,m),\phi(x',m')]
\end{aligned}$$

Hence ϕ is a morphism of Leibniz algebra and the two extensions are equivalents.

$\square$

Proposition 1.3.12. *Two abelian extensions $0 \to M \to \mathfrak{g}\oplus_\omega \to \mathfrak{g} \to 0$ and $0 \to M \to \mathfrak{g}\oplus_{\omega'} \to \mathfrak{g} \to 0$ are equivalent if and only if ω and ω' are cohomologous.*

Proof : Suppose there exists a morphism $\phi : \mathfrak{g} \oplus_\omega M \to \mathfrak{g} \oplus_{\omega'} M$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "id"'] & \mathfrak{g} \oplus_\omega M \arrow[r] \arrow[d, "\phi"] & \mathfrak{g} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] & \mathfrak{g} \oplus_{\omega'} M \arrow[r, "p"] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

ϕ is necessarily of the form $\phi(x,m) = (x, m + \alpha(x))$ where α is a linear map from $\mathfrak{g}$ to M. Moreover, we have

$$\begin{aligned}\phi([(x,m),(x',m')]) &= \phi([x,x'],[x,m']_L + [m,x']_R + \omega(x,x')) \\ &= ([x,x'],[x,m']_L + [m,x']_R + \omega(x,x') + \alpha([x,x']))\end{aligned}$$

and

$$\begin{aligned}[\phi(x,m),\phi(x',m')] &= [(x,m+\alpha(x)),(x',m'+\alpha(x'))] \\ &= ([x,x'],[x,m']_L + [x,\alpha(x')]_L + [m,x']_R + [\alpha(x),x']_R + \omega'(x,x'))\end{aligned}$$

Thus the fact that ϕ is a morphism of Leibniz algebra involves

$$\omega(x,x') - \omega'(x,x') = [x,\alpha(x')]_L + [\alpha(x),x']_R - \alpha([x,x'])$$

that is $\omega - \omega' = d_L\alpha$. Hence ω and ω' are cohomologous.

Conversely, if we suppose that ω and ω' are cohomologous, then there exists α such that $\omega - \omega' = d_L\alpha$. If we define $\phi : \mathfrak{g} \oplus_\omega M \to \mathfrak{g} \oplus_{\omega'} M$ by the formula $\phi(x,m) = (x, m + \alpha(x))$, then ϕ is a morphism of Leibniz algebra which makes the diagram commutative.

□

Hence we have the following theorem which links the set $Ext(\mathfrak{g}, M)$ of equivalence classes of abelian extensions of $\mathfrak{g}$ by M and the cohomology space $HL^2(\mathfrak{g}, M)$.

Theorem 1.3.13. *Let $\mathfrak{g}$ be a Leibniz algebra and let M be a $\mathfrak{g}$-module. Then there exists a bijection*

$$Ext(\mathfrak{g}, M) \simeq HL^2(\mathfrak{g}, M)$$

The characteristic abelian extension of a Leibniz algebra

Let $\mathfrak{g}$ be a Leibniz algebra, previously we have defined a Lie algebra $\mathfrak{g}_{Lie}$ canonically associated to $\mathfrak{g}$. We defined $\mathfrak{g}_{Lie}$ as the quotient of $\mathfrak{g}$ by the left ideal $\mathfrak{g}_{ann}$ generated by $\{[x,x], x \in \mathfrak{g}\}$. This ideal is precisely the obstruction for $\mathfrak{g}$ to be a Lie algebra.

Proposition 1.3.14. $[x,y] = 0 \ \ \forall x \in \mathfrak{g}_{ann}, y \in \mathfrak{g}$. *In particular $\mathfrak{g}_{ann}$ is abelian.*

Proof : This follows directly from Proposition 1.1.4.

□

Hence we have an abelian extension

$$\mathfrak{g}_{ann} \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_{Lie}$$

This extension is called the *characteristic abelian extension* of the Leibniz algebra $\mathfrak{g}$.

Because of Proposition 1.3.14, the induced structure of $\mathfrak{g}_{Lie}$-module on $\mathfrak{g}_{ann}$ is anti-symmetric. Thanks to Theorem 1.3.13, the class of this extension corresponds to a class of cohomology in $HL^2(\mathfrak{g}_{Lie}, \mathfrak{g}_{ann})$. We call this element, the *characteristic element* of the Leibniz algebra $\mathfrak{g}$.

The abelian extension by the left center of a Leibniz algebra

Let $\mathfrak{g}$ be a Leibniz algebra, the *left center* $Z_L(\mathfrak{g})$ is the kernel of the map $ad : \mathfrak{g} \to End(\mathfrak{g})$. That is,

$$Z_L(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [x, y] = 0 \,\forall y \in \mathfrak{g}\}$$

This is the kernel of a Leibniz algebra morphism, so it is an ideal of $\mathfrak{g}$. Hence, the quotient of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$ is a Leibniz algebra. Moreover, because of Proposition 1.3.14, we have $\mathfrak{g}_{ann} \subseteq Z_L(\mathfrak{g})$, thus the quotient of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$ is a Lie algebra. We denoted this quotient by g_0, this is a Lie subalgebra of $End(\mathfrak{g})$. $Z_L(\mathfrak{g})$ is an abelian Leibniz algebra, hence we have an abelian extension

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} g_0$$

This extension provides $Z_L(\mathfrak{g})$ with a structure of $\mathfrak{g}_0$-representation. By definition of $Z_L(\mathfrak{g})$, this is an anti-symmetric representation. This extension will be used in the integration of a Leibniz algebra.

The abelian extension by the center of a Leibniz algebra

Let $\mathfrak{g}$ be a Leibniz algebra, the *center* $Z(\mathfrak{g})$ is the subspace of $\mathfrak{g}$ defined by

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [x, y] = [y, x] = 0 \,\forall y \in \mathfrak{g}\}$$

Il is clearly an ideal of $\mathfrak{g}$, hence the quotient of $\mathfrak{g}$ by $Z(\mathfrak{g})$ is a Leibniz algebra. In this case, $\mathfrak{g}_{ann}$ is not necessarily included in $Z(\mathfrak{g})$, so the quotient $\mathfrak{g}_Z$ is not, a priori, a Lie algebra. The Leibniz algebra $Z(\mathfrak{g})$ is abelian, hence we have an abelian extension

$$Z(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_Z$$

This extension provides $Z(\mathfrak{g})$ with a structure of $\mathfrak{g}_Z$-representation. By definition of $Z(\mathfrak{g})$, this is a trivial representation.

1.3.4 A link between Leibniz cohomology with coefficients in a symmetric representation and Leibniz cohomology with coefficients in an antisymmetric representation

Let $\mathfrak{g}$ be a Leibniz algebra and let M be a vector space with a linear map $[-, -] : \mathfrak{g} \otimes M \to M$ which satisfies the axiom (LLM). We establish for all $n \in \mathbb{N}$ an isomorphism from the n-th cohomology group $HL^n(\mathfrak{g}, M^a)$ to the $(n-1)$-th cohomology group $HL^{n-1}(\mathfrak{g}, Hom(\mathfrak{g}, M)^s)$.

First, we have to define a structure of a symmetric $\mathfrak{g}$-representation on $Hom(\mathfrak{g}, M)$.

Proposition 1.3.15. *Let $\mathfrak{g}$ be a Leibniz algebra and let M be a vector space with a linear map $[-, -] : \mathfrak{g} \otimes M \to M$ which satisfies the axiom (LLM). Then the linear map $\{-, -\} : \mathfrak{g} \otimes Hom(\mathfrak{g}, M) \to M$ defined by*

$$\{x, \sigma\}(y) = [x, \sigma(y)] - \sigma([x, y])$$

satisfies (LLM).

Proof : Let $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}\{[x, y], \sigma\}(z) &= [[x, y], \sigma(z)] - \sigma([[x, y], z]) \\ &= [x, [y, \sigma(z)]] - [y, [x, \sigma(z)]] - \sigma([x, [y, z]]) + \sigma([y, [x, z]])\end{aligned}$$

and

$$\begin{aligned}\{x,\{y,\sigma\}\}(z)-\{y,\{x,\sigma\}\}(z) &= [x,\{y,\sigma\}(z)]-\{y,\sigma\}([x,z])-[y,\{x,\sigma\}(z)]+\{x,\sigma\}([y,z]) \\ &= [x,[y,\sigma(z)]]-[x,\sigma([y,z])]-[y,\sigma([x,z])]+\sigma([y,[x,z]]) \\ &\quad -[y,[x,\sigma(z)]]+[y,\sigma([x,z])]+[x,\sigma([y,z])]-\sigma([x,[y,z]]) \\ &= [x,[y,\sigma(z)]]+\sigma([y,[x,z]])-[y,[x,\sigma(z)]]-\sigma([x,[y,z]])\end{aligned}$$

Hence $\{[x,y],\sigma\}=\{x,\{y,\sigma\}\}-\{y,\{x,\sigma\}\}$ and (LLM) is satisfied.

$\square$

If we have a vector space M with a bracket $[-,-]$ which satisfies (LLM) we may define two structures of $\mathfrak{g}$-representations on it. One is symmetric taking $[-,-]_L=-[-,-]_R=[-,-]$, and the other is anti-symmetric taking $[-,-]_L=[-,-], [-,-]_R=0$. We will denote the symmetric structure by M^s and the anti-symmetric by M^a.

The following proposition links the cohomology of $\mathfrak{g}$ with coefficients in M^a and the cohomology of $\mathfrak{g}$ with coefficients in $Hom(\mathfrak{g},M)^s$.

Proposition 1.3.16. *Let $\mathfrak{g}$ be a Leibniz algebra and let M be a vector space with a linear map $[-,-]:\mathfrak{g}\otimes M\to M$ which satisfies (LLM).*
We have an isomorphism of cochain complexes

$$CL^n(\mathfrak{g},M^a)\xrightarrow{\tau^n}CL^{n-1}(\mathfrak{g},Hom(\mathfrak{g},M)^s)$$

given by $\omega\mapsto\tau^n(\omega)$ where

$$\tau^n(\omega)(x_1,\dots,x_{n-1})(x_n)=\omega(x_1,\dots,x_n)$$

Proof : This morphism is clearly an isomorphism $\forall n\geq 0$. Moreover, we have

$$\begin{aligned} d_L\tau^n(\omega)(x_0,\dots,x_{n-1})(x_n) &= \sum_{i=0}^{n-2}(-1)^i[x_i,\tau^n(\omega)(x_0,\dots,\hat{x_i},\dots,x_{n-1})](x_n) \\ &\quad +(-1)^{n-1}[x_{n-1},\tau^n(\omega)(x_0,\dots,x_{n-2})](x_n) \\ &\quad +\sum_{0\leq i<j\leq n-1}(-1)^{i+1}\tau^n(\omega)(x_0,\dots,x_{j-1},[x_i,x_j],x_{j+1},\dots,x_{n-1})(x_n) \\ &= \sum_{i=0}^{n-1}(-1)^i([x_i,\omega(x_0,\dots,\hat{x_i},\dots,x_{n-1},x_n)]-\omega(x_0,\dots,\hat{x_i},\dots,x_{n-1},[x_i,x_n])) \\ &\quad +\sum_{0\leq i<j\leq n-1}(-1)^{i+1}\omega(x_0,\dots,x_{j-1},[x_i,x_j],x_{j+1},\dots,x_{n-1},x_n) \\ &= \sum_{i=0}^{n-1}(-1)^i[x_i,\omega(x_0,\dots,\hat{x_i},\dots,x_{n-1},x_n)] \\ &\quad +\sum_{0\leq i<j\leq n}(-1)^{i+1}\omega(x_0,\dots,x_{j-1},[x_i,x_j],x_{j+1},\dots,x_{n-1},x_n) \\ &= d_L\omega(x_0,\dots,x_{n-1},x_n) \\ &= \tau^{n+1}(d_L\omega)(x_0,\dots,x_{n-1})(x_n)\end{aligned}$$

Hence $\{\tau^n\}_{n\geq 0}$ is a morphism of cochain complexes.

$\square$

Remark 1.3.17. This proposition is a generalisation of a remark given by T. Pirashvili in section 2 - Proposition 2.1 of his article [Pir94]. This is also a particular case of the Corollary 2.21 in the Master's thesis of Benoît Jubin [Jub06].

Chapter 2

Lie racks

2.1 Racks

2.1.1 Definitions

Definition 2.1.1. *A **rack** is a set X with a product $\rhd : X \times X \to X$ such that $x \rhd \_ : X \to X$ is a bijection for all $x \in X$ and which satisfies the* rack identity

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z) \quad \forall x, y, z \in X$$

Sometimes we denote the map $x \rhd \_ : X \to X$ by c_x. The rack identity becomes

$$c_x \circ c_y = c_{c_x(y)} \circ c_x$$

and because of the invertibility of c_x, we can rewrite the rack identity

$$c_x \circ c_y \circ c_x^{-1} = c_{c_x(y)}$$

In other word the map $c : X \to Bij(X)$ sends the product $\rhd$ in X to the conjugation in the group $Bij(X)$. Because of this fact, we call the product $\rhd$ the *conjugation*.

Remark 2.1.2. The conjugation in a rack X is non associative, so we have to be careful with the position of the parentheses in an expression $x_1 \rhd \ldots \rhd x_n$. In the sequel, the expression $x_1 \rhd \ldots \rhd x_n$ is equal to $x_1 \rhd (x_2 \rhd (\ldots \rhd (x_{n-1} \rhd x_n) \ldots))$, that is $(c_{x_1} \circ c_{x_2} \circ \cdots \circ c_{x_{n-1}})(x_n)$.

Remark 2.1.3. Let X be a rack, then we deduce directly from the axioms that $(x \rhd x) \rhd y = x \rhd y$ for all $x, y \in X$, but we do not have necessarily $x \rhd x = x$ for all $x \in X$. A rack satisfying this condition is called a *quandle*.

Definition 2.1.4. *A **morphism of racks** is a function $f : X \to Y$ such that for all $x, y \in X$*

$$f(x \rhd y) = f(x) \rhd f(y)$$

Observe that the rack identity is equivalent to the condition that c_x is an automorphism of racks for all $x \in X$.

2.1.2 Examples

Example 2.1.5 (Group)**.** A group G is a rack by taking the conjugation as the product $\rhd$

$$g \rhd h := ghg^{-1}$$

The axioms are satisfied because $g \rhd \_ : G \to G$ is a bijection with inverse $g^{-1} \rhd \_$ and we have

$$g \rhd (h \rhd k) = ghkh^{-1}g^{-1} = ghg^{-1}gkg^{-1}gh^{-1}g^{-1} = (g \rhd h) \rhd (g \rhd k)$$

We denote this rack by $Conj(G)$, G_{conj} or G if there is no ambiguity. Moreover a group morphism gives rise to a rack morphism, hence we can define a faithfull functor $Conj$ from the category of groups to the category of racks.

Example 2.1.6 (Digroup)**.** The notion of digroup is a generalisation of the notion of group. This structure was suggested by the work of J.-L. Loday on dialgebras (cf. [Lod97] or [Lod01]). This structure comes naturaly when we try to generalize the functors linking the categories of groups, Lie algebras and associative algebras.

$$\begin{tikzcd} Group \arrow[r, shift left, "\mathbb{Z}[-]"] & As \arrow[l, shift left, "(-)^*"] \arrow[r, shift left, "Lie"] & Lie \arrow[l, shift left, "U"] \end{tikzcd}$$

In the same way, we have functors linking the categories of digroups, Leibniz algebras ans dialgebras.

$$\begin{tikzcd} Digroup \arrow[r, shift left, "\mathbb{Z}[-]"] & Dias \arrow[l, shift left, "(-)^*"] \arrow[r, shift left, "Leib"] & Leib \arrow[l, shift left, "Ud"] \end{tikzcd}$$

Moreover, we have commutative diagrams of categories

$$\begin{tikzcd} As \arrow[r, "Lie"] \arrow[d, "inc"'] & Lie \arrow[d, "inc"] \\ Dias \arrow[r, "Leib"] & Leib \end{tikzcd}$$

and

$$\begin{tikzcd} Group \arrow[r, "\mathbb{Z}[-]"] \arrow[d, "inc"'] & As \arrow[d, "inc"] \\ Digroup \arrow[r, "\mathbb{Z}[-]"] & Dias \end{tikzcd}$$

Digroups were introduced and studied by several people at the same time (M.K. Kinyon [Kin07], K. Liu [Liu] and R. Felipe [Fel]). Here we give the definition, a brief summary of the properties of a digroup and the link with racks.

Definition 2.1.7. *A **digroup** is a set X with two associative products $\vdash, \dashv : X \times X \to X$ and a distinguished element $1 \in X$ which satisfy*

1.

$$(x \vdash y) \vdash z = (x \dashv y) \vdash z$$
$$x \dashv (y \dashv z) = x \dashv (y \vdash z)$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z)$$

2. for all $x \in X$, $1 \vdash x = x \dashv 1 = x$.

3. for all $x \in X$ there exists an element $y \in X$ such that $x \vdash y = y \dashv x = 1$.

Example :

1. A group is an example of a digroup where $\vdash = \dashv = \cdot$, the group multiplication.

2. Let G be a group and A a G-set with a fixed point 0, then there is a digroup structure on the cartesian product $G \times A$ putting

$$\begin{aligned}
(g, a) \vdash (h, b) &= (gh, g.b) \\
(g, a) \dashv (h, b) &= (gh, a) \\
(g, a)^{-1} &= (g^{-1}, 0) \\
1 &= (1, 0)
\end{aligned}$$

Remark 2.1.8. An element which verifies the third axiom is necessarily unique. Indeed, suppose that there exist $y, z \in X$ such that $y \dashv x = x \vdash y = z \dashv x = x \vdash z = 1$. We have

$$\begin{aligned}
z &= 1 \vdash z \\
&= (y \dashv x) \vdash z \\
&= (y \vdash x) \vdash z \\
&= y \vdash (x \vdash z) \\
&= y \vdash 1
\end{aligned}$$

If we exchange the roles of y and z, we find $y = z \vdash 1$, hence $y = z \vdash 1 = y \vdash 1 \vdash 1 = y \vdash 1$ and $z = y$. We denote this element by x^{-1}.

Proposition 2.1.9. *Let X be a digroup, we have*

1. $x^{-1} \vdash 1 = 1 \dashv x^{-1} = x^{-1}$

2. $x \vdash 1 = 1 \dashv x = (x^{-1})^{-1}$

3. $(x \vdash y)^{-1} = y^{-1} \vdash x^{-1} = y^{-1} \dashv x^{-1}$.

Proof :

1. This is clear, because of the remark above.

2. $x^{-1} \vdash (x \vdash 1) = (x^{-1} \vdash x) \vdash 1 = (x^{-1} \dashv x) \vdash 1 = 1 \vdash 1 = 1$ and $(x \vdash 1) \dashv x^{-1} = x \vdash (1 \dashv x^{-1}) = x \vdash x^{-1} = 1$.

3. (a) $(x \vdash y) \vdash (y^{-1} \vdash x^{-1}) = ((x \vdash y) \vdash y^{-1}) \vdash x^{-1} = (x \vdash 1) \vdash x^{-1} = 1$
   (b) $(y^{-1} \vdash x^{-1}) \dashv (x \vdash y) = (y^{-1} \vdash x^{-1}) \dashv (x \dashv y) = y^{-1} \vdash (x^{-1} \dashv (x \dashv y)) = y^{-1} \vdash (1 \dashv y) = (y^{-1} \vdash 1) \dashv y = y^{-1} \dashv y = 1$
   (c) $(x \vdash y) \vdash (y^{-1} \dashv x^{-1}) = ((x \vdash y) \vdash y^{-1}) \dashv x^{-1} = (x \vdash 1) \dashv x^{-1} = x \vdash (1 \dashv x^{-1}) = x \vdash x^{-1} = 1$
   (d) $(y^{-1} \dashv x^{-1}) \dashv (x \vdash y) = (y^{-1} \dashv x^{-1}) \dashv (x \dashv y) = y^{-1} \dashv (x^{-1} \dashv (x \dashv y)) = y^{-1} \dashv (1 \dashv y) = (y^{-1} \dashv 1) \dashv y = y^{-1} \dashv y = 1$

□

Let X be a digroup, then there are two remarkable subsets of X denoted by $I(X)$ and $E(X)$, defined by

$$I(X) = \{x^{-1} \in X \mid x \in X\}$$

and

$$E(X) = \{e \in X \mid e \vdash x = x \dashv e = x \quad \forall x \in X\}$$

Proposition 2.1.10. *Let X be a digroup, we have*

1. *$I(X)$ is a group.*
2. *$E(X) = \{x^{-1} \vdash x \in X \mid x \in X\} = \{x \dashv x^{-1} \in X \mid x \in X\}$.*
3. *$I(X)$ acts on $E(X)$.*
4. *for all $x \in X$, there exists a unique couple $(y^{-1}, e) \in I(X) \times E(X)$ such that $x = y^{-1} \vdash e$.*

Proof :

1. Let $x^{-1}, y^{-1} \in I(X)$, we put

$$x^{-1} \cdot y^{-1} = x^{-1} \vdash y^{-1} (= x^{-1} \dashv y^{-1})$$

We have $1 = 1^{-1}$ so $1 \in I(X)$. Moreover $1 \cdot x^{-1} = x^{-1} \cdot 1 = x^{-1}$ and $(x^{-1})^{-1} \cdot x^{-1} = x^{-1} \cdot (x^{-1})^{-1} = 1$, because of the preceding results.
Hence $I(X)$ is a group.

2. Let $x \in X$, we show first that $x^{-1} \vdash x \in E(X)$.
Let $y \in X$, we have $(x^{-1} \vdash x) \vdash y = (x^{-1} \dashv x) \vdash y = 1 \vdash y = y$ and $y \dashv (x^{-1} \vdash x) = y \dashv (x^{-1} \dashv x) = y \dashv 1 = y$.
Hence $x^{-1} \vdash x \in E(X)$.
Now let $e \in E(X)$, we have $e \vdash 1 = 1 \dashv e = 1$ so $e^{-1} = 1$.
So $e = e^{-1} \vdash e$ and $E(X) = \{x^1 \vdash x \in X / x \in X\}$.
In the same way, one shows that $E(X) = \{x \dashv x^{-1} \in X \mid x \in X\}$.

3. We define an action of $I(X)$ on $E(X)$ putting

$$x^{-1}.(y^{-1} \vdash y) = x^{-1} \vdash (y^{-1} \vdash y) \dashv (x^{-1})^{-1}$$

We have $x^{-1}.(y^{-1} \vdash y) \in E(X)$ because $(x^{-1}.(y^{-1} \vdash y)) \vdash z = (x^{-1} \vdash (y^{-1} \vdash y) \dashv (x^{-1})^{-1}) \vdash z = (x^{-1} \vdash (y^{-1} \vdash y) \vdash (x^{-1})^{-1}) \vdash z = z$ and $z \dashv (x^{-1}.(y^{-1} \vdash y)) = z \dashv (x^{-1} \vdash (y^{-1} \vdash y) \dashv (x^{-1})^{-1}) = z \dashv (x^{-1} \dashv (y^{-1} \vdash y) \dashv (x^{-1})^{-1}) = z$. A routine calculation shows that $x^{-1}.(y^{-1}.(z^{-1} \vdash z)) = (x^{-1}y^{-1}).(z^{-1} \vdash z)$.

4. Let $x \in X$, we have $x = (x^{-1})^{-1} \vdash (x^{-1} \vdash x)$. Hence there exist $y^{-1} \in I(X)$ and $e \in E(X)$ such that $x = y^{-1} \vdash e$. Moreover, if we suppose that there exists a couple (y^{-1}, e) such that $x = y^{-1} \vdash e$, then $x^{-1} = (y^{-1} \vdash e)^{-1} = e^{-1} \vdash (y^{-1})^{-1} = (y^{-1})^{-1}$, so $(x^{-1})^{-1} = y^{-1}$. On the other hand, we have $x^{-1} \vdash x = (y^{-1})^{-1} \vdash (y^{-1} \vdash e) = e$. Hence the couple $(y^{-1}, e) \in I(X) \times E(X)$ which verifies $x = y^{-1} \vdash e$ is necessarily unique.

□

Corollary 2.1.11. *The product $x \rhd y := x \vdash y \dashv x^{-1}$ satisfies the rack identity.*

Proof : We have

$$
\begin{aligned}
(x \rhd y) \rhd (x \rhd z) &= (((x \vdash y) \dashv x^{-1}) \vdash (x \vdash (z \dashv x^{-1}))) \dashv (x \vdash (y \dashv x^{-1}))^{-1} \\
&= (((x \vdash y) \vdash x^{-1}) \vdash (x \vdash (z \dashv x^{-1}))) \dashv (x \vdash (y^{-1} \dashv x^{-1})) \\
&= ((((x \vdash y) \vdash x^{-1}) \vdash x) \vdash (z \dashv x^{-1})) \dashv (x \vdash (y^{-1} \dashv x^{-1})) \\
&= ((((x \vdash y) \vdash x^{-1}) \vdash x) \vdash (z \dashv x^{-1})) \dashv (x \vdash (y^{-1} \dashv x^{-1})) \\
&= (((x \vdash y) \vdash (x^{-1} \dashv x)) \vdash (z \dashv x^{-1})) \dashv (x \vdash (y^{-1} \dashv x^{-1})) \\
&= (((x \vdash y) \vdash 1) \vdash (z \dashv x^{-1})) \dashv (x \vdash (y^{-1} \dashv x^{-1})) \\
&= ((x \vdash y) \vdash (1 \vdash (z \dashv x^{-1}))) \dashv (x \vdash (y^{-1} \dashv x^{-1})) \\
&= ((x \vdash y) \vdash (z \dashv x^{-1})) \dashv (x \vdash (y^{-1} \dashv x^{-1})) \\
&= (x \vdash y) \vdash ((z \dashv x^{-1}) \dashv (x \vdash (y^{-1} \dashv x^{-1}))) \\
&= (x \vdash y) \vdash ((z \dashv x^{-1}) \dashv (x \vdash (y^{-1} \dashv x^{-1}))) \\
&= (x \vdash y) \vdash (((z \dashv x^{-1}) \dashv x) \vdash (y^{-1} \dashv x^{-1})) \\
&= (x \vdash y) \vdash ((z \dashv (x^{-1} \dashv x)) \vdash (y^{-1} \dashv x^{-1})) \\
&= (x \vdash y) \vdash ((z \dashv 1) \vdash (y^{-1} \dashv x^{-1})) \\
&= (x \vdash y) \vdash (z \vdash (y^{-1} \dashv x^{-1})) \\
&= x \vdash (y \vdash (z \vdash y^{-1})) \dashv x^{-1} \\
&= x \rhd (y \rhd z)
\end{aligned}
$$

$\square$

Corollary 2.1.12. *If $(X, \vdash, \dashv, 1)$ is a digroup, then $(X, \rhd)$ is a rack.*

Example 2.1.13 (Augmented rack)**.** This is maybe the most important example of rack. Indeed, any rack might be provided with the structure of an augmented rack. In this example, the rack structure comes from a group action on a set, and an augmented rack is not so far from a crossed module (see [FR, FRS95]).

Let G be a group, let X be a G-set and a function $X \xrightarrow{p} G$ satisfying the *augmentation identity*, that is for all $g \in G$ and $x \in X$

$$p(g.x) = gp(x)g^{-1}.$$

If we consider G acting on itself by conjugation and on X, this condition is just the equivariance of p with respect to these actions. Then we can define a rack structure on X putting

$$x \rhd y := p(x).y$$

For all $x \in X$, the map $x \rhd \_ : X \to X$ is a bijection, because this map is defined by the action of G on X. Moreover, the rack identity is true because of the augmentation identity. Indeed, for all $x, y, z \in X$ we have

$$\begin{aligned} x \triangleright (y \triangleright z) &= p(x).(p(y).z) \\ &= (p(x)p(y)).z \\ &= (p(x)p(y)p(x)^{-1}p(x)).z \\ &= (p(x)p(y)p(x)^{-1}).(p(x).z) \\ &= p(p(x).y).(p(x).z) \\ &= (x \triangleright y) \triangleright (x \triangleright z) \end{aligned}$$

We call $X \xrightarrow{p} G$ an *augmented rack.*

Example :

1. A group G can be viewed as the augmented rack $G \xrightarrow{id} G$, where G acts on itself by conjugation.
2. Let X be a digroup, then $X \xrightarrow{i} I(X)$ is an example of an augmented rack. Indeed, $I(X)$ is a group and it acts on X by

   $$x^{-1}.y = x^{-1} \vdash y \dashv (x^{-1})^{-1}$$

   We have $i(x^{-1}.y) = i(x^{-1} \vdash y \dashv (x^{-1})^{-1}) = x^{-1} \vdash y^{-1} \vdash (x^{-1})^{-1} = x^{-1} \triangleright i(y)$, thus i is equivariant.

Definition 2.1.14. *Let $X \xrightarrow{p} G$ and $Y \xrightarrow{q} H$ two augmented racks. A* morphism of augmented racks *$f : (X \xrightarrow{p} G) \to (Y \xrightarrow{q} H)$ is a pair of maps (f_u, f_d) where*

1. *The map $f_d : G \to H$ is a morphism of groups.*
2. *$f_u : X \to Y$ satisfies $f_u(g.x) = f_d(g).f_u(x)$ for all $g \in G$ and $x \in X$.*
3. *The following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "f_u"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ G \arrow[r, "f_d"] & H \end{tikzcd}$$

Proposition 2.1.15. *Let $f : (X \xrightarrow{p} G) \to (Y \xrightarrow{q} H)$ be a morphism of augmented racks, then $f_u : X \to Y$ is a morphism of racks.*

2.1.3 Some groups associated to racks

Free group: Let X be a rack, then the free group generated by the set X, $F(X)$, acts on X by

$$(x_1 \dots x_n).x = (c_{x_1} \circ \dots \circ c_{x_n})(x)$$

Bijection group: Let X be a rack, then the group of bijections of X, $Bij(X)$, acts on X by

$$\phi.x = \phi(x)$$

Automorphism group: Let X be a rack, then the group of automorphims of X, $Aut(X)$, is a subgroup of $Bij(X)$ and so acts on X.

Proposition 2.1.16. *$X \xrightarrow{c} Aut(X)$ is an augmented rack.*

Proof : $Aut(X)$ is a group and acts on X, so the only thing that we have to verify is the equivariance of c, that is $c_{\phi(x)} = \phi \circ c_x \circ \phi^{-1}$. We have

$$\begin{aligned} c_{\phi(x)}(\phi(y)) &= \phi(x) \triangleright \phi(y) \\ &= \phi(x \triangleright y) \\ &= (\phi \circ c_x)(y) \end{aligned}$$

Hence $c_{\phi(x)} \circ \phi = \phi \circ c_x$, that is, c is equivariant.

□

Operator group: Let X be a rack, the *operator group of* X, $Op(X)$, is the subgroup of $Aut(X)$ generated by the image of $c : X \to Bij(X)$. $Op(X)$ is a subgroup of $Aut(X)$, so it acts on X and $X \xrightarrow{c} Op(X)$ is an augmented rack.

Associated group: Let X be a rack, the *associated group of* X (or *universal enveloping group*), $As(X)$, is the quotient of the free group $F(X)$ by the normal subgroup generated by the set $\{(xy^{-1}x^{-1})(x \triangleright y) \,|\, x, y \in X\}$.

Proposition 2.1.17. *The action of $F(X)$ on X induces an action of $As(X)$ on X, and the natural map $X \xrightarrow{\mu} As(X)$ is an augmented rack.*

Proof : $As(X)$ is a group and acts on X, so the only thing that we have to verify is the equivariance of μ, that is $\mu(w.x) = w\mu(x)w^{-1}$. Let $x_1 \dots x_n$ a representative of the class w. We have

$$\begin{aligned} \mu(w.x) &= \mu((c_{x_1} \circ \dots \circ c_{x_n})(x)) \\ &= \mu(x_1)\mu((c_{x_2} \circ \dots \circ c_{x_n})(x))\mu({x_1}^{-1}) \\ &= \dots \\ &= \mu(x_1) \dots \mu(x_n)\mu(x)\mu({x_n}^{-1}) \dots \mu({x_1}^{-1}) \\ &= (x_1 \dots x_n)\mu(x)({x_n}^{-1} \dots {x_1}^{-1}) \end{aligned}$$

Hence μ is equivariant.

□

The importance of $As(X)$ comes from the fact that this group satisfies the following universal property.

Proposition 2.1.18. *Let X be a rack and let G be a group. Given any morphism of racks $f : X \to Conj(G)$, there exists a unique morphism of groups $f_\natural : As(X) \to G$ which makes the following diagram commute*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & As(X) \arrow[d, "f_\natural"] \\ Conj(G) \arrow[r, "id"'] & G \end{tikzcd}$$

Moreover, any group with the same universal property is isomorphic to $As(X)$.

Proof : Let $f : X \to Conj(G)$ be a morphism of racks, then there exists a unique group morphism $\varphi : F(X) \to G$. Moreover, the fact that f is a morphism of racks implies that $\varphi((x \rhd y)xy^{-1}x^{-1}) = 1$ for all $x, y \in X$. Hence the morphism φ factors to a morphism $f_\natural : As(X) \to G$, and the commutativity of the diagram is clear. The uniqueness of $As(X)$ follows by the usual universal property argument.

$\square$

The following corollary is an easy consequence of this proposition.

Corollary 2.1.19. *As is left adjoint to Conj.*

Finally, we have seen that an augmented rack $X \xrightarrow{p} G$ induces a rack structure on X. Conversely, any rack X might be seen as an augmented rack with structure group $As(X)$. That is, we have a forgetful functor *AugmentedRack* $\to$ *Rack*, which has a left adjoint *Rack* $\to$ *AugmentedRack*.

2.1.4 Pointed racks

In the case of groups, the neutral element 1 has certain properties which play an important role for the link between Lie groups and Lie algebras. Indeed, to define a Lie algebra structure on the tangent space at 1 of a Lie group, we use the property $g \rhd 1 = 1$ forall $g \in G$ in order to define the morphism *Ad*. Likewise, we use the property $1 \rhd g = g, \forall g \in G$ in order to define the morphism *ad*. Hence, our goal being to extract the necessary and sufficient properties of the Lie group structure which permit us to associate a Lie algebra to it, we have to take into account those properties. These properties lead us to the concept of *pointed rack*.

Definition 2.1.20. *A rack is called* ***pointed****, if there exists an element $1 \in X$ such that*

$$1 \rhd x = x \text{ and } x \rhd 1 = 1 \quad \forall x \in X$$

Definition 2.1.21. *If X and Y are pointed, a morphism of pointed racks is a morphism of racks such that $f(1) = 1$.*

Example 2.1.22 (Group)**.** We have seen that a group G is a rack with the conjugation as product. This rack is pointed by $1 \in G$. Indeed, we have

$$1 \rhd g = g \text{ and } g \rhd 1 = 1 \quad \forall g \in G$$

Example 2.1.23 (Digroup)**.** We have seen that a digroup $(X, \vdash, \dashv)$ is a rack with the product

$$x \rhd y = x \vdash y \dashv x^{-1}$$

This rack is pointed by the neutral element of the products $\vdash$ and $\dashv$. Indeed, we have

$$1 \rhd x = 1 \vdash x \dashv 1 = x \text{ and } x \rhd 1 = x \vdash 1 \dashv x^{-1} = (x^{-1})^{-1} \vdash x^{-1} = 1 \quad \forall x \in X$$

Example 2.1.24 (Pointed augmented rack)**.** We have seen in Example 2.1.13 that, given an augmented rack $X \xrightarrow{p} G$, we have a rack structure on X given by

$$x \rhd y = p(x).y$$

If there exists an element $1 \in X$ such that $p(1) = 1$ and $g.1 = 1 \ \ \forall g \in G$, then the augmented rack $X \xrightarrow{p} G$ is pointed.

Remark 2.1.25. We have seen in section 2.1.3 that there is a functor $Conj : Group \to Rack$ with left adjoint $As : Rack \to Group$. Actually, for G a group, $Conj(G)$ is a pointed rack, and we have a functor $Conj_p : Group \to PointedRack$. This functor has a left adjoint $As_p : PointedRack \to Group$ defined on the objects by $As_p(X) = As(X)/ < \{[1]\} >$, where $< \{[1]\} >$ is the subgroup generated by [1].

2.1.5 Topological, smooth and Lie racks

To generalize Lie groups, we need pointed racks with a differentiable structure compatible with the algebraic structure. This is the notion of Lie racks. We will see in Proposition 3.1.1 that a Lie rack provides the tangent space at the neutral element with the structure of a Leibniz algebra.

Topological racks

Definition 2.1.26. *A **topological rack** is a topological space X with a rack structure such that*

1. *The product $\rhd : X \times X \to X$ is continuous.*
2. *$\forall x \in X, c_x$ is a homeomorphism.*

*A topological rack X is **pointed** if X is a pointed rack.*

Smooth and Lie racks

Definition 2.1.27. *A **smooth rack** is a smooth manifold X with a rack structure such that*

1. *The product $\rhd : X \times X \to X$ is smooth.*
2. *$\forall x \in X, c_x$ is a diffeomorphism.*

*A **Lie rack** is a pointed smooth rack.*

2.1.6 Local racks

A Lie rack structure on a set is a global structure, i.e. defined on the whole set. In fact, to define a Leibniz algebra structure on the tangent space at a point, we only need a local Lie rack structure in a neighborhood of this point. This leads us to the concept of local racks. We will see in Proposition 3.1.5 that the tangent space at the neutral element of a local Lie rack is provided with the structure of a Leibniz algebra. The main result of our thesis is that conversely, to every Leibniz algebra, there exists a local Lie rack which integrates it.

Definitions

Definition 2.1.28. *A **local rack** is a set X with a product $\rhd$ defined on a subset Ω of $X \times X$ with values in X such that the following axioms are satisfied:*

1. *If $(x,y),(x,z),(y,z),(x,y\rhd z),(x\rhd y, x\rhd z) \in \Omega$, then $x\rhd(y\rhd z) = (x\rhd y)\rhd(x\rhd z)$.*
2. *If $(x,y),(x,z) \in \Omega$ and $x\rhd y = x\rhd z$, then $y = z$.*

Definition 2.1.29. *Let X and Y two local racks, a **morphism of local racks** is a map $f : X \to Y$ such that if $x\rhd y$ is defined in X, then $f(x)\rhd f(y)$ is defined and equal to $f(x\rhd y)$.*

Examples

Example 2.1.30. Let X a rack, then every subset U of X is a local rack.

Example 2.1.31 (Local group)**.** (see W.T. Van Est[Est62] or N. Bourbaki [Bou72])

Definition 2.1.32. *A local group is a set G with a product m, defined on a subset Ω of $G \times G$ and with values in G, and a map $i : G \to G$ such that the following axioms are satisfied*

1. *If $(g,h),(h,k),(m(g,h),k),(g,m(h,k)) \in \Omega$ then $m(m(g,h),k) = m(g,m(h,k))$.*
2. *$\forall g \in G, (1,g),(g,1) \in \Omega$ and $m(1,g) = m(g,1) = g$.*
3. *$\forall g \in G, (i(g),g),(g,i(g)) \in \Omega$ and $m(i(g),g) = m(g,i(g)) = 1$.*

Let G a local group, then G is a local rack putting $x\rhd y = xyx^{-1}$ whenever this expression is defined.

2.1.7 Pointed local racks

Definition

Definition 2.1.33. *A local rack is called **pointed**, if there exists a distinguished element $1 \in X$ such that $1\rhd x$ and $x\rhd 1$ are defined for all $x \in X$ and are respectively equal to x and 1. This element is called the **neutral element**.*

Examples

Example 2.1.34. Let X a rack, then we have stated that every subset U of X is a local rack. Moreover, if we suppose that X is pointed and $1 \in U$, then U is a pointed local rack.

Example 2.1.35 (Local group)**.** We have seen that a local group G is a local rack. This is also a pointed local rack with neutral element $1 \in G$.

2.1.8 Topological, smooth and Lie local racks

Definition 2.1.36. *A **topological local rack** is a topological space X with the stucture of a local rack with respect to a subset $\Omega \in X \times X$ such that*

1. *Ω is an open subset of $X \times X$.*
2. *$\rhd : \Omega \to X$ is continuous.*

*A topological local rack X is **pointed**, if X is pointed.*

Definition 2.1.37. *A **smooth local rack** is a smooth manifold X with a structure of local rack with respect to a subset $\Omega \in X \times X$ such that*

1. *Ω is an open subset of $X \times X$.*
2. *$\rhd : \Omega \to X$ is smooth.*

*A **local Lie rack** is a pointed smooth local rack.*

2.2 Rack modules

To define a cohomology theory for (pointed) racks, we first have to find the right definition of a (pointed) rack module. Here we take the definition given by N. Jackson [Jac07] which generalizes the definitions given by P. Etingof and M. Graña [EG03] and the more general one given by Andruskiewitsch and Graña [AG03].

2.2.1 Definition

Definition 2.2.1. *Let X be a rack, an X-**module** $\mathcal{A} = (A, \phi, \psi)$ is a family of abelian groups $\{A_x\}_{x \in X}$ with two families of homomorphisms of abelian groups $\phi_{x,y} : A_y \to A_{x \rhd y}$ and $\psi_{x,y} : A_x \to A_{x \rhd y}$ such that for all $x, y, z \in X$:*

(M_0) *$\phi_{x,y}$ is an isomorphism.*

(M_1) $\phi_{x,y \rhd z} \circ \phi_{y,z} = \phi_{x \rhd y, x \rhd z} \circ \phi_{x,z}$

(M_2) $\phi_{x,y \rhd z} \circ \psi_{y,z} = \psi_{x \rhd y, x \rhd z} \circ \phi_{x,y}$

(M_3) $\psi_{x,y \rhd z} = \phi_{x \rhd y, x \rhd z} \circ \psi_{x,z} + \psi_{x \rhd y, x \rhd z} \circ \psi_{x,y}$

*The X-modules where all the A_x are isomorphic, are called **homogeneous**, and those where this is not the case, are called **heterogeneous**. A homogeneous X-module where $A_x = A$, $\phi_{x,y} = id$ and $\psi_{x,y} = 0$ for all $x, y \in X$ is said to be trivial.*

Remark 2.2.2. Let G be a group, a G-module is the same as a functor $\mathcal{A} : C(G) \to Ab$, where $C(G)$ is the category associated to G with one object and one morphism for each element of G. Given a rack X, we can define an X-module in the same way. In this case, the construction does not yield a category, but a so-called trunk. We gather more information about trunks in appendix A.

Let X be a rack, we define a trunk $T(X)$ by setting

Objects: $x \in X$

Morphisms: for all $x, y \in X$, two morphisms $\alpha_{x,y} : A_y \to A_{x \rhd y}$ and $\beta_{x,y} : A_x \to A_{x \rhd y}$

Prefered squares:

$$\begin{tikzcd} z \arrow[r, "\alpha_{y,z}"] \arrow[d, "\alpha_{x,z}"'] & y \rhd z \arrow[d, "\alpha_{x,y \rhd z}"] \\ x \rhd z \arrow[r, "\alpha_{x \rhd y, x \rhd z}"'] & x \rhd y \rhd z \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "\beta_{y,z}"] \arrow[d, "\alpha_{x,y}"'] & y \rhd z \arrow[d, "\alpha_{x,y \rhd z}"] \\ x \rhd y \arrow[r, "\beta_{x \rhd y, x \rhd z}"'] & x \rhd y \rhd z \end{tikzcd}$$

We easily see that an X-module $\mathcal{A}$ is a trunk map $\mathcal{A} : T(X) \to Ab$ such that the axioms (M_0) and (M_3) are satisfied (see Appendix A for the definition of trunks).

2.2.2 Examples

Example 2.2.3. Let X be a rack and let A be an abelian group, then A is canonically a homogeneous trivial X-module.

Example 2.2.4. Let X be a rack and let A be an abelian group equipped with an action of $As(X)$. We define a homogeneous X-module putting for all $x, y \in X$

$$A_x = A$$

$$\phi_{x,y}(a) = x.a$$
$$\psi_{x,y}(a) = a - (x \triangleright y).a$$

We call this X-module the **symmetric** X-module on A and denote it by A^s.

Example 2.2.5. Let X be a rack and let A be an abelian group equipped with an action of $As(X)$. We define a homogeneous X-module putting for all $x, y \in X$

$$A_x = A$$

$$\phi_{x,y}(a) = x.a$$
$$\psi_{x,y}(a) = 0$$

We call this X-module the **anti-symmetric** X-module on A and denote it by A^a.

Remark 2.2.6. The trivial module A, denoted by A^{tr}, is an example of a symmetric and anti-symmetric module. In this case, the action of $As(X)$ is trivial.

2.2.3 Pointed rack module

2.2.4 Definition

Definition 2.2.7. *Let X be a pointed rack, an X-**pointed module** $\mathcal{A} = (A, \phi, \psi)$ is an X-module which satisfies the following axiom*

(M_4) $\phi_{1,y} = id_{A_y}$ $\forall y \in X$ *and* $\psi_{x,1} = 0$ $\forall x \in X$

2.2.5 Examples

Example 2.2.8. Let X be a pointed rack and let A be an abelian group, then A is canonically a homogeneous trivial X-pointed module.

Example 2.2.9. Let X be a pointed rack and let A be an abelian group equipped with an action of $As(X)$. We have seen that A^s is an X-module. In fact, A^s is a pointed X-module, because we have

$$\phi_{1,y}(a) = 1.a = a \text{ and } \psi_{x,1}(a) = a - (x \triangleright 1).a = a - 1.a = 0$$

Example 2.2.10. Let X be a pointed rack and let A be an abelian group equipped with an action of $As(X)$. We have seen that A^a is an X-module. In fact, in the same way as in the preceding example, A^a is a pointed X-module, because we have

$$\phi_{1,y}(a) = 1.a = a \text{ and } \psi_{x,1}(a) = 0$$

Remark 2.2.11. The trivial module is an example of a symmetric and anti-symmetric pointed module, where the action of $As(X)$ is trivial. We denote a pointed module A with a trivial structure still by A^{tr}.

2.3 Cohomology

It is natural to ask if, like for many algebraic structures (groups, Lie algebras, associative algebras), there exists a cohomology theory associated to (pointed) racks. In this section, we expose the definition given by N. Jackson in [Jac] which generalizes the definition given first by P. Etingof and M. Graña [EG03] and secondly by N. Andruskiewitsch and M. Graña [AG03]. Then we show that the second cohomology group is in bijection with the set of equivalence classes of abelian extensions.

A cohomology theory being defined for (pointed) racks and a group being a (pointed) rack, another natural question is whether there exists a link between the cohomology of groups and the cohomology of racks. The explicit link is explained in Proposition 2.3.24.

We finish this section with the definition of smooth and local cohomology for racks. The link between rack cohomology and Leibniz cohomology will be given in the next section.

2.3.1 Cohomology of racks

Definition

Let X be a rack and let $\mathcal{A} = (A, \phi, \psi)$ be an X-module. We define a cochain complex $\{CR^n(X, \mathcal{A}), d_R^n\}_{n\geq 0}$ putting

$$CR^n(X, \mathcal{A}) = \{\{f_{x_1,\ldots,x_n} \in A_{x_1\rhd\ldots\rhd x_n}\}_{(x_1,\ldots,x_n)\in X^n}\}$$

and

$$d_R^n : CR^n(X, \mathcal{A}) \to CR^{n+1}(X, \mathcal{A})$$

where

$$\begin{aligned}(d_R^n f)_{x_1,\ldots,x_{n+1}} &= \sum_{i=1}^{n}(-1)^{i-1}(\phi_{x_1\rhd\ldots\rhd x_i, x_1\rhd\ldots\rhd\widehat{x_i}\rhd\ldots\rhd x_{n+1}}(f_{x_1,\ldots,\widehat{x_i},\ldots,x_{n+1}}) - f_{x_1,\ldots,x_i\rhd x_{i+1},\ldots,x_i\rhd x_{n+1}}) \\ &\quad +(-1)^n\psi_{x_1\rhd\ldots\rhd x_n, x_1\rhd\ldots\rhd x_{n-1}\rhd x_{n+1}}(f_{x_1,\ldots,x_n})\end{aligned}$$

Remark 2.3.1. This expression is well-defined using the Lemma 2.3.20.

Lemma 2.3.2. $d_R^{n+1} \circ d_R^n = 0$

Proof : We decompose $d_R^n = \sum_{i=1}^{n+1}(-1)^{i-1}d_i^n$, where for $i \leq n$,

$$d_i^n f(x_1, \ldots, x_{n+1}) = \phi_{x_1\rhd\ldots\rhd x_i, x_1\rhd\ldots\rhd\widehat{x_i}\rhd\ldots\rhd x_{n+1}}((f_{x_1,\ldots,\widehat{x_i},\ldots,x_{n+1}}) - f_{x_1,\ldots,x_i\rhd x_{i+1},\ldots,x_i\rhd x_{n+1}})$$

and for $i = n + 1$,

$$d_{n+1}^n f(x_1, \ldots, x_{n+1}) = -\psi_{x_1\rhd\ldots\rhd x_n, x_1\rhd\ldots\rhd x_{n-1}\rhd x_{n+1}}(f_{x_1,\ldots,x_n})$$

Then, it is straightforward to verify that for $1 \leq i < j \leq n + 1$

$$d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n$$

and thus $d_R^{n+1} \circ d_R^n = 0$.

□

Definition 2.3.3. *Let X be a rack and let $\mathcal{A}$ be an X-module. The cohomology of X with coefficients in $\mathcal{A}$ is the cohomology of the cochain complex $\{CR^n(X, \mathcal{A}), d_R^n\}_{n\geq 0}$.*

$$HR^n(X, \mathcal{A}) := H^n(\{CR^n(X, \mathcal{A}), d_R^n\}_{n\geq 0}) \quad \forall n \geq 0$$

$HR^2(X,\mathcal{A})$ **and abelian extensions of racks**

Definition 2.3.4. *An (abelian)* ***extension*** *of the rack X by an X-module $\mathcal{A}=(A,\phi,\psi)$ is a rack epimorphism*

$$E \overset{p}{\twoheadrightarrow} X$$

which satisfies the following axioms.

(E_0) *for all $x\in X$, there is a simply transitively right action of A_x on $p^{-1}(x)$.*

(E_1) *for all $u\in p^{-1}(x), a\in A_x, v\in p^{-1}(y)$, we have $(u.a)\rhd v=(u\rhd v).\psi_{x,y}(a)$.*

(E_2) *for all $u\in p^{-1}(x), v\in p^{-1}(y), a\in A_y$, we have $u\rhd(v.a)=(u\rhd v).\phi_{x,y}(a)$.*

Definition 2.3.5. *Two extensions $E_1 \overset{p_1}{\twoheadrightarrow} X$ and $E_2 \overset{p_2}{\twoheadrightarrow} X$ of a rack X by an X-module $\mathcal{A}=(A,\phi,\psi)$ are called* ***equivalent****, if there exists a rack isomorphism*

$$E_1 \xrightarrow{\theta} E_2$$

which satisfies the following axioms

1. $p_2\circ\theta=p_1$.
2. $\forall x\in X, u\in(E_1)_x, a\in A_x$ *we have* $\theta(u.a)=\theta(u).a$.

We denote by $Ext(X,\mathcal{A})$ the set of equivalences classes of X by $\mathcal{A}$.

Example 2.3.6. Let X be a rack, let $\mathcal{A}=(A,\phi,\psi)$ be an X-module and $f\in ZR^2(X,\mathcal{A})$. Then we define an abelian extension of X by $\mathcal{A}$

$$E(\mathcal{A},f)\overset{p}{\twoheadrightarrow} X,$$

by

$$E(\mathcal{A},f)=\{(x,a)\,|\,a\in A_x, x\in X\},$$

with rack operation

$$(x,a)\rhd(y,b)=(x\rhd y,\phi_{x,y}(b)+\psi_{x,y}(a)+f_{x,y}),$$

where p is the projection onto the first factor. The action of A_x on $p^{-1}(x)$ is defined by $(x,a).b=(x,a+b)$. It is clear that this action satisfies the axiom (E_0).
This product satisfies the rack identity, because of the axioms $(M_1),(M_2),(M_3)$ and the cocycle identity. Indeed, we have

$$\begin{aligned}(x,a)\rhd((y,b)\rhd(z,c)) &= (x,a)\rhd(y\rhd z,\phi_{y,z}(c)+\psi_{y,z}(b)+f_{y,z})\\ &= (x\rhd(y\rhd z),\phi_{x,y\rhd z}(\phi_{y,z}(c)+\psi_{y,z}(b)+f_{y,z})+\psi_{x,y\rhd z}(a)+f_{x,y\rhd z})\\ &= (x\rhd(y\rhd z),(\phi_{x,y\rhd z}\circ\phi_{y,z})(c)+(\phi_{x,y\rhd z}\circ\psi_{y,z})(b)+\phi_{x,y\rhd z}(f_{y,z})\\ &\quad+\psi_{x,y\rhd z}(a)+f_{x,y\rhd z})\end{aligned}$$

$$
\begin{aligned}
((x,a)\rhd(y,b))\rhd((x,a)\rhd(z,c)) &= (x\rhd y, \phi_{x,y}(b)+\psi_{x,y}(a)+f_{x,y})\rhd(x\rhd z, \phi_{x,z}(c)+\psi_{x,z}(a)+f_{x,z}) \\
&= ((x\rhd y)\rhd(x\rhd z), \phi_{x\rhd y,x\rhd z}(\phi_{x,z}(c)+\psi_{x,z}(a)+f_{x,z}) \\
&+ \psi_{x\rhd y,x\rhd z}(\phi_{x,y}(b)+\psi_{x,y}(a)+f_{x,y})+f_{x\rhd y,x\rhd z}) \\
&= ((x\rhd y)\rhd(x\rhd z), (\phi_{x\rhd y,x\rhd z}\circ\phi_{x,z})(c)+(\phi_{x\rhd y,x\rhd z}\circ\psi_{x,z})(a) \\
&+ \phi_{x\rhd y,x\rhd z}(f_{x,z})+(\psi_{x\rhd y,x\rhd z}\circ\phi_{x,y})(b)+(\psi_{x\rhd y,x\rhd z}\circ\psi_{x,y})(a) \\
&+ \psi_{x\rhd y,x\rhd z}(f_{x,y})+f_{x\rhd y,x\rhd z})
\end{aligned}
$$

By the axioms of a module, we have

$$
\begin{aligned}
(M_1) &\Rightarrow \phi_{x,y\rhd z}\circ\phi_{y,z} = \phi_{x\rhd y,x\rhd z}\circ\phi_{x,z} \\
(M_2) &\Rightarrow \phi_{x,y\rhd z}\circ\psi_{y,z} = \psi_{x\rhd y,x\rhd z}\circ\phi_{x,y} \\
(M_3) &\Rightarrow \psi_{x,y\rhd z} = \phi_{x\rhd y,x\rhd z}\circ\psi_{x,z}+\psi_{x\rhd y,x\rhd z}\circ\psi_{x,y}
\end{aligned}
$$

and by the cocycle identity, we have

$$
\phi_{x,y\rhd z}(f_{y,z})+f_{x,y\rhd z} = \phi_{x\rhd y,x\rhd z}(f_{x,z})+f_{x\rhd y,x\rhd z}+\psi_{x\rhd y,x\rhd z}(f_{x,y})
$$

Hence the rack identity is satisfied.
Moreover, we have

$$
\begin{aligned}
((x,a).b)\rhd(y,c) &= (x,a+b)\rhd(y,c) \\
&= (x\rhd y, \phi_{x,y}(c)+\psi_{x,y}(a+b)+f_{x,y}) \\
&= (x\rhd y, \phi_{x,y}(c)+\psi_{x,y}(a)+\psi_{x,y}(b)+f_{x,y}) \\
&= (x\rhd y, \phi_{x,y}(c)+\psi_{x,y}(a)+f_{x,y}).\psi_{x,y}(b) \\
&= ((x,a)\rhd(y,c)).\psi_{x,y}(b)
\end{aligned}
$$

and

$$
\begin{aligned}
(x,a)\rhd((y,b).c)) &= (x,a)\rhd(y,b+c) \\
&= (x\rhd y, \phi_{x,y}(b+c)+\psi_{x,y}(a)+f_{x,y}) \\
&= (x\rhd y, \phi_{x,y}(b)+\phi_{x,y}(c)+\psi_{x,y}(a)+f_{x,y}) \\
&= (x\rhd y, \phi_{x,y}(b)+\psi_{x,y}(a)+f_{x,y}).\phi_{x,y}(c) \\
&= ((x,a)\rhd(y,b)).\phi_{x,y}(c)
\end{aligned}
$$

Thus the axioms (E_1) and (E_2) are satisfied, and $E(\mathcal{A},f) \overset{p}{\twoheadrightarrow} X$ is an abelian extension of X by $\mathcal{A}$.

Proposition 2.3.7. *Let X be a rack and let $\mathcal{A}$ be an X-module. Every equivalence class of abelian extensions in $Ext(X,\mathcal{A})$ can be represented by an abelian extension of the form $E(\mathcal{A},f) \overset{p}{\twoheadrightarrow} X$.*

Proof : Let $E \overset{\pi}{\twoheadrightarrow} X$ be an abelian extension of X by $\mathcal{A}$ and let s be a section of π. We have for all $x \in X$ $s(x) \in \pi^{-1}(x)$, because of $(\pi\circ s)(x) = x$, so by the axiom (E_1), for all $u \in E_x = \pi^{-1}(x)$, there exists a unique $a \in A_x$ such that $u = s(x).a$.
Since π is a homomorphism, $s(x)\rhd s(y) \in \pi^{-1}(x\rhd y)$, and there exists a unique $f_{x,y} \in A_{x\rhd y}$ such that $s(x)\rhd s(y) = s(x\rhd y).f_{x,y}$.

We have

$$\begin{aligned} s(x)\triangleright(s(y)\triangleright s(z)) &= s(x)\triangleright(s(y\triangleright z).f_{y,z}) \\ &= (s(x)\triangleright s(y\triangleright z)).\phi_{x,y\triangleright z}(f_{y,z}) \\ &= (s(x\triangleright(y\triangleright z)).f_{x,y\triangleright z}).\phi_{x,y\triangleright z}(f_{y,z}) \\ &= s(x\triangleright(y\triangleright z)).(f_{x,y\triangleright z}) + \phi_{x,y\triangleright z}(f_{y,z}) \end{aligned}$$

and

$$\begin{aligned} (s(x)\triangleright s(y))\triangleright(s(x)\triangleright s(z)) &= (s(x\triangleright y).f_{x,y})\triangleright(s(x\triangleright z).f_{x,z}) \\ &= (s(x\triangleright y)\triangleright(s(x\triangleright z).f_{x,z})).\psi_{x\triangleright y,x\triangleright z}(f_{x,y}) \\ &= ((s(x\triangleright y)\triangleright s(x\triangleright z)).\phi_{x\triangleright y,x\triangleright z}(f_{x,z})).\psi_{x\triangleright y,x\triangleright z}(f_{x,y}) \\ &= ((s((x\triangleright y)\triangleright(x\triangleright z)).f_{x\triangleright y,x\triangleright z}).\phi_{x\triangleright y,x\triangleright z}(f_{x,z})).\psi_{x\triangleright y,x\triangleright z}(f_{x,y}) \\ &= (s(x\triangleright(y\triangleright z))).(f_{x\triangleright y,x\triangleright z}) + \phi_{x\triangleright y,x\triangleright z}(f_{x,z})) + \psi_{x\triangleright y,x\triangleright z}(f_{x,y}) \end{aligned}$$

Moreover, by the axiom (E_0), we have

$$f_{x,y\triangleright z} + \phi_{x,y\triangleright z}(f_{y,z}) = f_{x\triangleright y,x\triangleright z} + \phi_{x\triangleright y,x\triangleright z}(f_{x,z})) + \psi_{x\triangleright y,x\triangleright z}(f_{x,y})$$

and this is exactly the cocycle identity. Hence $f \in ZR^2(X, \mathcal{A})$.
Now we want to show that this extension is equivalent to $E(\mathcal{A}, f) \overset{p}{\twoheadrightarrow} X$, that is, we have to find an isomorphism $E(\mathcal{A}, f) \xrightarrow{\theta} E$ such that

$$\begin{aligned} \pi \circ \theta &= p \\ \theta((x, a).b) &= \theta(x, a).b \end{aligned}$$

We take

$$\theta(x, a) = s(x).a$$

θ is clearly an isomorphism with inverse $\theta^{-1}(u) = (\pi(u), a)$, where a is the unique element in $A_{\pi(u)}$ such that $u = s(\pi(u)).a$.
Moreover, we have

$$\pi(\theta(x, a)) = \pi(s(x).a) = x = p(x)$$

and

$$\theta((x, a).b) = \theta(x, a + b) = s(x).(a + b) = (s(x).a).b = (\theta(x, a)).b$$

Hence $E(\mathcal{A}, f) \xrightarrow{\theta} E$ is an equivalence.

□

Proposition 2.3.8. *Two abelian extensions $E(\mathcal{A}, f_1) \overset{p_1}{\twoheadrightarrow} X$ and $E(\mathcal{A}, f_2) \overset{p_2}{\twoheadrightarrow} X$ are equivalent if and only if f_1 and f_2 are cohomologous.*

Proof : Suppose there exists an isomorphism $E(\mathcal{A}, f_1) \xrightarrow{\theta} E(\mathcal{A}, f_2)$ such that

$$\begin{aligned} p_2 \circ \theta &= p_1 \\ \theta((x, a).b) &= \theta(x, a).b, \end{aligned}$$

then for all $x \in X$, there exists $\sigma_x \in A_x$ such that $\theta(x, 0) = (x, \sigma_x)$. Moreover, we have

$$\theta(x, a) = \theta((x, 0).a) = \theta(x, 0).a = (x, \sigma_x).a = (x, \sigma_x + a)$$

for all $a \in A_x$, since θ preserves the right actions of A_x.
Furthermore

$$\theta((x,a)\triangleright(y,b)) = \theta(x\triangleright y, \phi_{x,y}(b) + \psi_{x,y}(a) + (f_1)_{x,y}) = (x\triangleright y, \phi_{x,y}(b) + \psi_{x,y}(a) + (f_1)_{x,y} + \sigma_{x\triangleright y})$$

and

$$\theta(x,a)\triangleright\theta(y,b) = (x,\sigma_x + a)\triangleright(y,\sigma_y + b) = (x\triangleright y, \phi_{x,y}(b) + \phi_{x,y}(\sigma_y) + \psi_{x,y}(a) + \psi_{x,y}(\sigma_x) + (f_2)_{x,y})$$

are equal, because θ is a rack isomorphism, and so

$$(f_1)_{x,y} - (f_2)_{x,y} = \phi_{x,y}(\sigma_y) - \sigma_{x\triangleright y} + \psi_{x,y}(\sigma_x)$$

Hence f_1 and f_2 are cohomologous.

Conversely, if we suppose that f_1 and f_2 are cohomologous, then there exists σ such that $f_1 - f_2 = d_R\sigma$, and if we define $\theta : E(\mathcal{A}, f_1) \to E(\mathcal{A}, f_2)$ by the formula $\theta(x,a) = (x, a + \sigma_x)$, then θ is a isomorphism of racks. This shows that the two extensions are equivalent.

□

Because of the preceding propositions, we have the following theorem which relates the set of equivalence classes $Ext(X,\mathcal{A})$ and $HR^2(X,\mathcal{A})$.

Theorem 2.3.9. *Let X be a rack and let $\mathcal{A}$ be an X-module. Then there exists a bijection*

$$Ext(X,\mathcal{A}) \simeq HR^2(X,\mathcal{A})$$

2.3.2 Cohomology of pointed racks

Definition

Let X be a pointed rack and let $\mathcal{A} = (A, \phi, \psi)$ be an X-pointed module. We define a cochain complex $\{CR^n_p(X,\mathcal{A}), d^n_R\}_{n\geq 0}$ putting

$$CR^n_p(X,\mathcal{A}) = \{\{f_{x_1,\dots,x_n} \in A_{x_1\triangleright\dots\triangleright x_n}\}_{(x_1,\dots,x_n)\in X^n} \mid f_{x_1,\dots,1,\dots,x_n} = 0\}$$

and

$$d^n_R : CR^n(X,\mathcal{A}) \to CR^{n+1}(X,\mathcal{A})$$

where

$$\begin{aligned}(d^n_R f)_{x_1,\dots,x_{n+1}} &= \sum_{i=1}^{n}(-1)^{i-1}(\phi_{x_1\triangleright\dots\triangleright x_i, x_1\triangleright\dots\triangleright\widehat{x_i}\triangleright\dots\triangleright x_{n+1}}(f_{x_1,\dots,\widehat{x_i},\dots,x_{n+1}}) - f_{x_1,\dots,x_i\triangleright x_{i+1},\dots,x_i\triangleright x_{n+1}}) \\ &\quad +(-1)^n\psi_{x_1\triangleright\dots\triangleright x_n, x_1\triangleright\dots\triangleright x_{n-1}\triangleright x_{n+1}}(f_{x_1,\dots,x_n})\end{aligned}$$

Lemma 2.3.10. $d^{n+1}_R \circ d^n_R = 0$

Proof : Same as in Lemma 2.3.2

□

Definition 2.3.11. *Let X be a pointed rack and let $\mathcal{A}$ be an X-pointed module. The cohomology of X with coefficients in $\mathcal{A}$ is the cohomology of the cochain complex $\{CR^n_p(X,\mathcal{A}), d^n_R\}_{n\geq 0}$.*

$$HR^n_p(X,\mathcal{A}) := H^n(\{CR^n_p(X,\mathcal{A}), d^n_R\}_{n\geq 0}) \quad \forall n \geq 0$$

$HR^2_p(X,\mathcal{A})$ and abelian extensions of pointed racks

Definition 2.3.12. *An **extension** of the pointed rack X by an X-pointed module $\mathcal{A}=(A,\phi,\psi)$ is an extension of the rack X by the X-module $\mathcal{A}$*

$$E \overset{p}{\twoheadrightarrow} X$$

such that p is a morphism of pointed racks.

Definition 2.3.13. *Two extensions $E_1 \overset{p_1}{\twoheadrightarrow} X$ and $E_2 \overset{p_2}{\twoheadrightarrow} X$ of a pointed rack X by an X-pointed module $\mathcal{A}=(A,\phi,\psi)$ are **equivalent** if there exist an isomorphism of pointed rack*

$$E_1 \overset{\theta}{\to} E_2$$

We denote by $Ext_p(X,\mathcal{A})$ the set of equivalences classes of X by M.

Example 2.3.14. Let X be a pointed rack, let $\mathcal{A}=(A,\phi,\psi)$ be an X-pointed module and $f\in ZR^2_p(X,\mathcal{A})$. Then we define an abelian extension of X by $\mathcal{A}$

$$E_p(\mathcal{A},f) \overset{p}{\twoheadrightarrow} X$$

where

$$E_p(\mathcal{A},f)=\{(x,a)\,|\,a\in A_x, x\in X\},$$

with rack operation

$$(x,a)\rhd(y,b)=(x\rhd y,\phi_{x,y}(b)+\psi_{x,y}(a)+f_{x,y}),$$

neutral element $(1,0)$ and p the projection on the first factor. The action of A_x on $p^{-1}(x)$ is defined by $(x,a).b=(x,a+b)$. It is clear that this action satisfied the axiom (E_0).
We just have to verify that $(1,0)$ is a neutral element and p is a pointed rack morphism, for the rest, this is the same raisonnement as before.
The map p is clearly a pointed rack momorphism, and because of (M_4) and $f_{x,1}=0, f_{1,x}=0$, we have

$$\begin{aligned}(1,0)\rhd(y,b)&=(1\rhd y,\phi_{1,y}(b)+\psi_{1,y}(0)+f_{1,y})=(y,b)\\(x,a)\rhd(1,0)&=(x\rhd 1,\phi_{x,1}(0)+\psi_{x,1}(a)+f_{x,1})=(x,a)\end{aligned}$$

Proposition 2.3.15. *Let X be a pointed rack and let $\mathcal{A}$ be an X-pointed module. Every class of abelian extensions in $Ext_p(X,\mathcal{A})$ can be represented by an abelian extension of the form $E_p(\mathcal{A},f) \overset{p}{\twoheadrightarrow} X$.*

Proof : Let $E \overset{\pi}{\twoheadrightarrow} X$ be an abelian extension of X by $\mathcal{A}$ and let s be a section of π such that $s(1)=1$. We construct the cocycle f in the same way as in (2.3.7). Furthermore $s(y)=s(1)\rhd s(y)=s(1\rhd y).f_{1,y}=s(y).f_{1,y}$ and $1=s(x)\rhd s(1)=s(x\rhd 1).f_{x,1}=1.f_{x,1}$ so $f_{1,y}=0$ and $f_{x,1}=0$. Hence $f\in ZR^2_p(X,\mathcal{A})$.
Moreover, we construct an equivalence between $E_p(\mathcal{A},f)$ and E using exactly the same proof as (2.3.7).

□

Proposition 2.3.16. *Two abelian extensions $E_p(\mathcal{A}, f_1) \overset{p_1}{\twoheadrightarrow} X$ and $E_p(\mathcal{A}, f_2) \overset{p_2}{\twoheadrightarrow} X$ are equivalent if and only if f_1 and f_2 are cohomologous.*

Proof : This is the same proof as before. We just have to verify that σ satisfies $\sigma_1 = 1$. Since $\theta(1,0) = (1, \sigma_1) = (1, 0)$, we have $\sigma_1 = 1$.

□

Because of the preceding propositions, we have the following theorem which links the set of equivalence classes $Ext_p(X, \mathcal{A})$ and $HR^2_p(X, \mathcal{A})$.

Theorem 2.3.17. *Let X be a rack and let $\mathcal{A}$ be an X-module. Then there exists a bijection*

$$Ext_p(X, \mathcal{A}) \simeq HR^2_p(X, \mathcal{A})$$

2.3.3 Cohomology with coefficients in an $As(X)$-module

This is our most important example. In this part, we link the cohomology of a rack in a symmetric module to the cohomology of a rack in an anti-symmetric module. This result is the analogue in the category of racks of Proposition 1.3.16 in the category of Leibniz algebras.

Let X be a (pointed) rack and let A be an abelian group with the structure of an $As(X)$-module. We recall that we can put on A two structures of an X-(pointed) rack module. The first structure is symmetric, denoted A^s, and given by

$$\begin{aligned} A^s_x &= A \\ \phi_{x,y}(a) &= x.a \\ \psi_{x,y}(a) &= a - (x \rhd y).a \end{aligned}$$

for all $x, y \in X, a \in A$.
The second is anti-symmetric, denoted A^a, and given by

$$\begin{aligned} A^a_x &= A \\ \phi_{x,y}(a) &= x.a \\ \psi_{x,y}(a) &= 0 \end{aligned}$$

for all $x, y \in X, a \in A$.
In Proposition 1.3.15 , we have shown that if M is a module over a Leibniz algebra $\mathfrak{g}$, then $Hom(\mathfrak{g}, M)$ is a $\mathfrak{g}$-module too. The following proposition is the analoguous result for modules over a rack.

Proposition 2.3.18. *Let X be a rack and let A be an $As(X)$-module, then $C^1(X, A) = Map(X, A)$ is an $As(X)$-module.*

Proof : A is an $As(X)$-module, i.e. there exists a map $\phi : X \to Bij(A)$ such that for all $x, y \in X$

$$\phi_{x \rhd y} \circ \phi_x = \phi_x \circ \phi_y$$

We define a map $\Phi : X \to Bij(C^1(X, A))$ by putting $\forall x \in X, f \in C^1(X, A)$

$$\Phi_x(f) = \phi_x \circ f \circ c_x^{-1}$$

It is clear that Φ_x is a bijection with inverse Φ_x^{-1} defined by

$$\Phi_x^{-1}(f) = \phi_x^{-1} \circ f \circ c_x$$

Now let $x, y \in X$ and $f \in C^1(X, M)$, we have

$$\begin{aligned}
\Phi_{x \rhd y}(\Phi_x(f)) &= \phi_{x \rhd y} \circ \Phi_x(f) \circ c_x^{-1} \\
&= \phi_{x \rhd y} \circ \phi_x \circ f \circ c_x^{-1} \circ c_{x \rhd y}^{-1} \\
&= \phi_x \circ \phi_y \circ f \circ c_y^{-1} \circ c_x^{-1} \\
&= \Phi_x(\Phi_y(f))
\end{aligned}$$

Hence Φ satisfies the identity $\Phi_x(\Phi_y(f)) = \Phi_{x \rhd y}(\Phi_x(f))$.

□

The following proposition establishes, like in the Leibniz algebra case, an isomorphism between the n-th cohomology group of a rack with coefficients in a symmetric module A^s and the $(n-1)$-th cohomology group of a rack in the antisymmetric module $C^1(X, A)^a$.

Proposition 2.3.19. *Let X be a rack and let A be an $As(X)$-module. We have an isomorphism of cochain complexes*

$$CR^n(X, A^s) \xrightarrow{\sigma_n} CR^{n-1}(X, C^1(X, A)^a)$$

given by

$$\sigma_n(f)(x_1, \dots, x_{n-1})(x_n) = f(x_1, \dots, x_{n-1}, (c_{x_{n-1}}^{-1} \circ \dots \circ c_{x_1}^{-1})(x_n))$$

To prove this proposition, we need some computational lemma.

Lemma 2.3.20. *Let X be a rack. We have the following identity*

$$(x_1 \rhd \dots \rhd x_i) \rhd (x_1 \rhd \dots \rhd \widehat{x_i} \rhd \dots \rhd x_n \rhd x) = x_1 \rhd \dots \rhd x_n \rhd x$$

where $\widehat{x_i}$ denotes the omission of the factor x_i in the product.

Proof : We have

$$\begin{aligned}
(x_1 \rhd \dots \rhd x_i) \rhd (x_1 \rhd \dots \rhd \widehat{x_i} \rhd \dots \rhd x_n \rhd x) &= x_1 \rhd ((x_2 \rhd \dots \rhd x_i) \rhd (x_2 \rhd \dots \rhd \widehat{x_i} \rhd \dots \rhd x_n \rhd x) \\
&= x_1 \rhd x_2 \rhd ((x_3 \rhd \dots \rhd x_i) \rhd (x_3 \rhd \dots \rhd \widehat{x_i} \rhd \dots \rhd x_n \rhd x) \\
&= x_1 \rhd \dots \rhd x_{i-1} \rhd (x_i \rhd (x_{i+1} \rhd \dots \rhd x_n \rhd x)) \\
&= x_1 \rhd \dots \rhd x_n \rhd x
\end{aligned}$$

□

Lemma 2.3.21. *Let X be a rack. We have the following identity*

$$x_1 \rhd \dots \rhd x_{i-1} \rhd (x_i \rhd x_{i+1}) \rhd \dots \rhd (x_i \rhd x_n) \rhd (x_i \rhd x) = x_1 \rhd \dots \rhd x_n \rhd x$$

Proof : We have

$$\begin{aligned}
x_1 \rhd \dots \rhd x_{i-1} \rhd (x_i \rhd x_{i+1}) \rhd \dots \rhd (x_i \rhd x_n) \rhd (x_i \rhd x) &= x_1 \rhd \dots \rhd x_{i-1} \rhd (x_i \rhd x_{i+1}) \rhd \dots \rhd (x_i \rhd x_n \rhd x) \\
&= x_1 \rhd \dots \rhd x_{i-1} \rhd (x_i \rhd x_{i+1}) \rhd \dots \rhd (x_i \rhd x_{n-1} \rhd x_n \rhd x) \\
&= x_1 \rhd \dots \rhd x_{i-1} \rhd (x_i \rhd x_{i+1} \rhd \dots \rhd x_n \rhd x) \\
&= x_1 \rhd \dots \rhd x_n \rhd x
\end{aligned}$$

$\square$

Proof of the proposition: It is clear that σ_n is a bijection for all $n \in \mathbb{N}$ with inverse σ_n^{-1} defined by

$$\sigma_n^{-1}(f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{n-1}, (c_{x_1} \circ \cdots \circ c_{x_{n-1}})(x_n))$$

Now let $f \in CR^n(X, M)$ and $x_1, \ldots, x_{n+1} \in X$. By definition,

$$\begin{aligned} d_R\sigma_n(f)(x_1, \ldots, x_n)(x_{n+1}) = \sum_{i=1}^{n} (-1)^{i-1} \big(&\Phi_{x_1 \triangleright \ldots \triangleright x_i}(\sigma_n(f)(x_1, \ldots, \widehat{x_i}, \ldots, x_n))(x_{n+1}) \\ &- \sigma_n(f)(x_1, \ldots, x_i \triangleright x_{i+1}, \ldots, x_i \triangleright x_n)(x_{n+1}) \big). \end{aligned}$$

We have

$$\Phi_{x_1 \triangleright \ldots \triangleright x_i}(\sigma_n(f)(x_1, \ldots, \widehat{x_i}, \ldots, x_n)) = \phi_{x_1 \triangleright \ldots \triangleright x_i} \circ \sigma_n(f)(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \circ c^{-1}_{x_1 \triangleright \ldots \triangleright x_i}$$

and

$$\sigma_n(f)(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \circ c^{-1}_{x_1 \triangleright \ldots \triangleright x_i} = f(x_1, \ldots, \widehat{x_i}, \ldots, x_n, (c^{-1}_{x_n} \circ \ldots \widehat{c^{-1}_{x_i}} \cdots \circ c^{-1}_{x_1}) \circ (c^{-1}_{x_1 \triangleright \ldots \triangleright x_i})(\_)).$$

Because of Lemma 2.3.20, we have

$$c_{x_1 \triangleright \ldots \triangleright x_i} \circ c_{x_1} \circ \cdots \circ \widehat{c_{x_i}} \circ \cdots \circ c_{x_n} = c_{x_1} \circ \cdots \circ c_{x_n},$$

so

$$(c^{-1}_{x_n} \circ \ldots \widehat{c^{-1}_{x_i}} \cdots \circ c^{-1}_{x_1}) \circ c^{-1}_{x_1 \triangleright \ldots \triangleright x_i} = (c_{x_1} \circ \cdots \circ c_{x_n})^{-1} = c^{-1}_{x_n} \circ \cdots \circ c^{-1}_{x_1},$$

hence we have

$$(\sigma_n(f)(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \circ c^{-1}_{x_1 \triangleright \ldots \triangleright x_i})(x_{n+1}) = f(x_1, \ldots, \widehat{x_i}, \ldots, x_n, (c^{-1}_{x_n} \circ \cdots \circ c^{-1}_{x_1})(x_{n+1})),$$

and

$$\Phi_{x_1 \triangleright \ldots \triangleright x_i}(\sigma_n(f)(x_1, \ldots, \widehat{x_i}, \ldots, x_n))(x_{n+1}) = \phi_{x_1 \triangleright \ldots \triangleright x_i}(f(x_1, \ldots, \widehat{x_i}, \ldots, x_n, (c^{-1}_{x_n} \circ \cdots \circ c^{-1}_{x_1})(x_{n+1})).$$

Moreover,

$$\sigma_n(f)(x_1, \ldots, x_i \triangleright x_{i+1}, \ldots, x_i \triangleright x_n) = f(x_1, \ldots, x_i \triangleright x_{i+1}, \ldots, x_i \triangleright x_n, c^{-1}_{x_i \triangleright x_n} \circ \cdots \circ c^{-1}_{x_i \triangleright x_{i+1}} \circ \cdots \circ c^{-1}_{x_1}).$$

Because of Lemma 2.3.21, we have

$$c_{x_1} \circ \cdots \circ c_{x_i \triangleright x_{i+1}} \circ \cdots \circ c_{x_i \triangleright x_n} = c_{x_1} \circ \cdots \circ c_{x_n} \circ c^{-1}_{x_i},$$

so

$$c^{-1}_{x_i \triangleright x_n} \circ \cdots \circ c^{-1}_{x_i \triangleright x_{i+1}} \circ \cdots \circ c^{-1}_{x_1} = c_{x_i} \circ c^{-1}_{x_n} \circ \cdots \circ c^{-1}_{x_1}.$$

Hence, we deduce

$$\sigma_n(f)(x_1, \ldots, x_i \triangleright x_{i+1}, \ldots, x_i \triangleright x_n)(x_{n+1}) = f(x_1, \ldots, x_i \triangleright x_{i+1}, \ldots, x_i \triangleright x_n, (c_{x_i} \circ c^{-1}_{x_n} \circ \cdots \circ c^{-1}_{x_1})(x_{n+1})).$$

On the other hand, we have

$$\begin{aligned} \sigma_{n+1}(d_R f)(x_1, \ldots, x_n)(x_{n+1}) &= d_R f(x_1, \ldots, x_n, (c^{-1}_{x_n} \circ \cdots \circ c^{-1}_{x_1})(x_{n+1})) \\ &= \sum_{i=1}^{n} (-1)^{i-1} (\phi_{x_1 \triangleright \ldots \triangleright x_i}(f(x_1, \ldots, \widehat{x_i}, \ldots, (c^{-1}_{x_n} \circ \cdots \circ c^{-1}_{x_1})(x_{n+1})) \\ &\quad - f(x_1, \ldots, x_i \triangleright x_{i+1}, \ldots, x_i \triangleright x_n, (c_{x_i} \circ c^{-1}_{x_n} \circ \cdots \circ c^{-1}_{x_1})(x_{n+1}))). \end{aligned}$$

Hence, we get

$$d_R \circ \sigma_n = \sigma_{n+1} \circ d_R,$$

and $\{\sigma_n\}_{n\in\mathbb{N}}$ is a morphism of cochains complexes.

$\square$

2.3.4 Link with group cohomology

The structure of a pointed rack on a group G comes from the group structure on G, so it is natural to hope for a morphism between the group cohomology and the pointed rack cohomology of G. We show here that there exists such a morphism. We will see in Proposition 3.3.6 that this morphism derives to the morphism defined in Proposition 1.3.4 between the Lie cohomology and the Leibniz cohomology of a Lie algebra.

Cohomology of groups

First we recall the definition of group cohomology (cf [CE56],[HS97], [Mac95]). Let G be a group and let A be a G-module, we define a cochain complex putting

$$C^n(G,A) = \{f : G^n \to A / f(g_1,\dots,1,\dots,g_n) = 0\}$$

and

$$d^n : C^n(X,A) \to C^{n+1}(X,A)$$

where

$$(d^n f)(g_1,\dots,g_{n+1}) \quad = g_1.f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_{n+1}) + (-1)^{n+1} f(g_1,\dots,g_n)$$

Definition 2.3.22. *Let G be a group and let A be a G-module. The group cohomology of G with coefficients in A is the cohomology of the cochain complex*

$$H^n(G,A) := H^n(\{C^n(G,A), d^n\}_{n\geq 0})$$

Group module and rack module

Next, to compare the two cohomologies, we have to compare the structures of modules in the two cases.

Proposition 2.3.23. *Let G be a group and let A be a G-module, then A is an $As(Conj(G))$-module.*

Proof : This is clear by adjointness. Indeed, that As is left adjoint to $Conj$ means that there exists a natural bijection

$$Hom_{Rack}(X, Conj(G)) \simeq Hom_{Group}(As(X), G)$$

A group morphism $G \to Bij(A)$ induces a morphism of racks $Conj(G) \to Conj(Bij(A))$, and by adjointness, there exists a unique group morphism $As(Conj(G)) \to Bij(A)$.

$\square$

A morphism from group cohomology to rack cohomology

Proposition 2.3.24. *Let G be a group and A a G-module. We have a morphism of cochain complexes*

$$C^n(G,A) \xrightarrow{\Delta^n} CR^n_p(G,A^s)$$

defined recursively by

$$\Delta^0 = id$$

$$\Delta^n(f)(g_1,\dots,g_n) = \sum_{i=1}^{n}(-1)^{i-1}\Delta^{n-1}(f(g_1 \triangleright \dots \triangleright g_i))(g_1,\dots,\widehat{g_i},\dots,g_n)$$

where $f(h_1)(h_2,\dots,h_n) = f(h_1,\dots,h_n)$.

An explicit formula for Δ^n is given by

$$\Delta^n(f)(g_1,\dots,g_n) = \sum_{\sigma\in\mathfrak{S}_n}(-1)^{sign(\sigma)}f(h_1^\sigma,\dots,h_n^\sigma)$$

where $h_k^\sigma = g_{j_1} \triangleright \dots \triangleright g_{j_l} \triangleright g_{\sigma(k)}$ with $j_1 < \dots < j_l < \sigma(k)$ and $j_i \notin \{\sigma(1),\dots,\sigma(k)\}$.

Proof : This proposition will be shown in Proposition A.7.9 using the simplicial description of these cohomology theories.

□

$\mathbf{n = 0}$: We have

$$C^0(G,A) = CR^0_p(G,A^s) = A$$
$$d^0(a)(g) = d^0_R(a)(g) = g.a$$
$$\Delta^0 = id$$

Hence $H^0(G,A) \simeq HR^0_p(G,A^s)$.

Remark 2.3.25. This morphism differentiates to an isomorphism $H^0(\mathfrak{g},A) \simeq HL^0(\mathfrak{g},A^s)$ for $\mathfrak{g}$ a Lie algebra.

$\mathbf{n = 1}$: We have

$$C^1(G,A) = CR^1_p(G,A^s)$$
$$d^1(f)(g_1,g_2) = g_1.f(g_2) - f(g_1g_2) + f(g_1)$$
$$d^1_R(f)(g_1,g_2) = g_1.f(g_2) - f(g_1 \triangleright g_2) - (g_1 \triangleright g_2).f(g_1) + f(g_1)$$
$$\Delta^1 = id$$

Hence $H^1(G,A) \overset{\Delta_1}{\hookrightarrow} HR^1_p(G,A^s)$.

Remark 2.3.26. This morphism differentiates to an isomorphism $H^1(\mathfrak{g},A) \simeq HL^1(\mathfrak{g},A^s)$ for $\mathfrak{g}$ a Lie algebra.

$\mathbf{n=2}$: We have

$$\begin{aligned}
C^2(G,A) &= CR^2_p(G,A^s)\\
d^2(f)(g_1,g_2,g_3) &= g_1.f(g_2,g_3) - f(g_1g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2)\\
d^2_R(f)(g_1,g_2,g_3) &= g_1.f(g_2,g_3) - f(g_1\rhd g_2, g_1\rhd g_3) - (g_1\rhd g_2).f(g_1,g_3) + f(g_1,g_2\rhd g_3)\\
&\quad + (g_1\rhd g_2\rhd g_3).f(g_1,g2) - f(g_1,g_2)\\
\Delta^2(f)(g_1,g_2) &= f(g_1,g_2) - f(g_1\rhd g_2, g_1)
\end{aligned}$$

Hence $H^2(G,A) \stackrel{\Delta^2}{\to} HR^2(G,A^s)$.

Remark 2.3.27. The map Δ^2 appears naturally when we write down the conjugation in an abelian extension of a group G by a module A. Indeed, Let G be a group, let A be a G-module and f an element of $Z^2(G,A)$. We put a group structure on $G\times_f A$ by setting $(g,a).(h,b) = (gh, a+g.b+f(g,h))$. We have $(g,a)^{-1} = (g^{-1}, -g^{-1}.a - f(g,g^{-1}))$, and the formula for the conjugation is (cf. [Nee04])

$$(g,a)\rhd(h,b) = (g\rhd h, a+g.b-(g\rhd h).a + f(g,h) - f(g\rhd h, g))$$

Hence $G\times_{\Delta^2(f)} A^s$ is a rack, that is, $\Delta^2(f)\in ZR^2(G,A^s)$.

2.3.5 Cohomology of Lie racks

Definition 2.3.28. *Let X be a Lie rack, a **smooth** X**-module** is a homogeneous X-pointed module (A,ϕ,ψ) such that*

1. *A is an abelian Lie group.*
2. *$\phi : X\times X\times A\to A$ is smooth.*
3. *$\psi : X\times X\times A\to A$ is smooth.*

Let X be a Lie rack and let A be a smooth X-module. We define a cochain complex $\{CR^n_p(X,A)_s, d^n_R\}_{n\geq 0}$ by taking for $CR^n_p(X,A)_s$ the subset of pointed rack cochains in $CR^n_p(X,A)$ which are smooth in a neighborhood of $(1,\ldots,1)$, regarded as functions from $X^n\to A$. For the coboundary operator d^n_R, we take the same formula as in Definition 2.3.11.

Definition 2.3.29. *Let X be a Lie rack and let A be a smooth X-module. Then the Lie rack cohomology of X with coefficients in A is the cohomology of the cochain complex $\{CR^n_p(X,A)_s, d^n_R\}_{n\geq 0}$.*

$$HR^n_s(X,A) := H^n(\{CR^n_p(X,A)_s, d^n_R\}_{n\geq 0}) \ \ \forall n\geq 0$$

2.3.6 Local cohomology

Definition 2.3.30. *Let U be a subset of a pointed rack X which contains 1, and let $\mathcal{A}$ be an X-pointed module such that $\psi\neq 0$. A **U-local** n**-tuple** is a n-tuple $(x_1,\ldots,x_n)\in X\times U^{n-1}$ such that the products $x_{i_1}\rhd\ldots\rhd x_{i_j}$ are in U, forall $i_1<\cdots<i_j, 2\leq j\leq n$.*
*If $\psi=0$, then a **U-local** n**-tuple** is a n-tuple $(x_1,\ldots,x_n)\in X^{n-1}\times U$ such that the products $x_{i_1}\rhd\ldots\rhd x_{i_j}\rhd x_n$ are in U, forall $i_1<\cdots<i_j<n, 1\leq j\leq n-1$.*
We denote by U_{n-loc} the set of U-local n-tuples.

Definition 2.3.31. *Let U be a subset of a pointed rack X and let $\mathcal{A}$ be an X-module. A* ***U-local*** ***n-cochain*** *is a family*

$$\{f_{x_1,\dots,x_n} \in A_{x_1 \triangleright \dots \triangleright x_n}\}_{(x_1,\dots,x_n)\in U_{n-loc}}$$

such that $f_{x_1,\dots,1,\dots,x_n} = 0$.
We denote by $CR^n_p(U,\mathcal{A})$ the set of U-local n-cochains.

Let X be a pointed rack, let U be a subset of X which contains 1 and let $\mathcal{A}$ a X-pointed module. We define a cochain complex $\{CR^n_p(U,\mathcal{A}), d^n_R\}_{n\geq 0}$ putting

$$\begin{aligned}(d^n_R f)_{x_1,\dots,x_{n+1}} &= \sum_{i=1}^{n}(-1)^{i-1}(\phi_{x_1\triangleright\dots\triangleright x_i, x_1\triangleright\dots\triangleright\widehat{x_i}\triangleright\dots\triangleright x_{n+1}}(f_{x_1,\dots,\widehat{x_i},\dots,x_{n+1}}) - f_{x_1,\dots,x_i\triangleright x_{i+1},\dots,x_i\triangleright x_{n+1}}) \\ &\quad +(-1)^n\psi_{x_1\triangleright\dots\triangleright x_n, x_1\triangleright\dots\triangleright x_{n-1}\triangleright x_{n+1}}(f_{x_1,\dots,x_n})\end{aligned}$$

Lemma 2.3.32. $d^{n+1}_R \circ d^n_R = 0$

Proof : Same as in Lemma 2.3.2.

□

Definition 2.3.33. *Let X be a pointed rack, let U be a subset of X which contains 1, and let $\mathcal{A}$ be an X-pointed module. The U-local cohomology of X with coefficients in $\mathcal{A}$ is the cohomology of the cochain complex $\{CR^n_p(U,\mathcal{A}), d^n_R\}_{n\geq 0}$.*

$$HR^n_p(U,\mathcal{A}) := H^n(\{CR^n_p(U,\mathcal{A}), d^n_R\}_{n\geq 0}) \quad \forall n \geq 0$$

Remark 2.3.34. Let X be a pointed rack. If we take $U = X$, we get the ordinary cohomology groups of X with coefficients in $\mathcal{A}$.

Chapter 3

Lie racks and Leibniz algebras

3.1 From Lie racks to Leibniz algebras

In this section we construct a functor from the category of Lie racks to the category of Leibniz algebras. The following proposition is in [Kin07].

Proposition 3.1.1. *Let X be a Lie rack, then T_1X is a Leibniz algebra.*

Proof : Let X be a Lie rack, we denote by $\mathfrak{x}$ the tangent space of X at 1. Recall that the conjugation $\rhd$ induces for all $x \in X$ an automorphism of Lie racks c_x (cf. Definition 2.1.1). We define for all $x \in X$

$$Ad_x = T_1c_x \in GL(\mathfrak{x})$$

and because of $c_{x\rhd y} = c_x \circ c_y \circ c_x^{-1}$ (cf. Definition 2.1.1) and $c_1 = id$ (cf. Definition 2.1.20), it gives us a morphism of Lie racks

$$Ad : X \to GL(\mathfrak{x}).$$

We can differentiate Ad at 1 and we obtain a linear map

$$ad : \mathfrak{x} \to \mathfrak{gl}(\mathfrak{x}).$$

Then we define a bracket $[-,-]$ on $\mathfrak{x} = T_1X$ by setting

$$[u, v] = ad(u)(v).$$

Now, we have to verify that this bracket satisfies the Leibniz identity, that is

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]].$$

To show this identity, we use the rack identity

$$x\rhd(y\rhd z) = (x\rhd y)\rhd(x\rhd z).$$

Let $u, v, w \in \mathfrak{x}$ and γ_u (resp. γ_v and γ_w) be a smooth path in X, such that $\gamma_u(0) = 1$ and $\left.\frac{\partial}{\partial s}\right|_{s=0}\gamma_u(s) = u$ (resp. $\gamma_v(0) = \gamma_w(0) = 1$, $\left.\frac{\partial}{\partial s}\right|_{s=0}\gamma_v(s) = v$ and $\left.\frac{\partial}{\partial s}\right|_{s=0}\gamma_w(s) = w$). We have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0}(\gamma_u(r)\rhd(\gamma_v(s)\rhd\gamma_w(t))) &= \left.\frac{\partial}{\partial t}\right|_{t=0}((c_{\gamma_u(r)} \circ c_{\gamma_v(s)})(\gamma_w(t)) \\ &= (Ad_{\gamma_u(r)} \circ Ad_{\gamma_v(s)})(w)\end{aligned}$$

and

$$\left.\frac{\partial}{\partial t}\right|_{t=0}((\gamma_u(r)\rhd\gamma_v(s))\rhd(\gamma_u(r)\rhd\gamma_w(t))) = \left.\frac{\partial}{\partial t}\right|_{t=0}((c_{\gamma_u(r)\rhd\gamma_v(s)}\circ c_{\gamma_u(r)})(\gamma_w(t))$$
$$= (Ad_{\gamma_u(r)\rhd\gamma_v(s)}\circ Ad_{\gamma_u(r)})(w)$$
$$= (Ad_{c_{\gamma_u(r)}(\gamma_v(s))}\circ Ad_{\gamma_u(r)})(w).$$

Moreover, if we differentiate with respect to the variable s

$$\left.\frac{\partial}{\partial s}\right|_{s=0}((Ad_{\gamma_u(r)}\circ Ad_{\gamma_v(s)})(w)) = (Ad_{\gamma_u(r)}\circ ad(v))(w)$$

and

$$\left.\frac{\partial}{\partial s}\right|_{s=0}(Ad_{c_{\gamma_u(r)}(\gamma_v(s))}\circ Ad_{\gamma_u(r)})(w) = (ad(Ad_{\gamma_u(r)}(v))\circ Ad_{\gamma_u(r)})(w).$$

And if we differentiate with respect to the variable r, we obtain

$$\left.\frac{\partial}{\partial r}\right|_{r=0}(Ad_{\gamma_u(r)}\circ ad(v))(w) = ad(u)(ad(v)(w))$$

and

$$\left.\frac{\partial}{\partial r}\right|_{r=0}(ad(Ad_{\gamma_u(r)}(v))\circ Ad_{\gamma_u(r)})(w) = ad(ad(u)(v))(w) + ad(v)(ad(u)(w))$$

Hence, by identification, we have the Leibniz identity.

□

Proposition 3.1.2. *Let $f : X \to Y$ be a morphism of Lie racks, then $T_1 f : T_1 X \to T_1 Y$ is a morphism of Leibniz algebras.*

Proof : The fact that f is a morphism of Lie rack implies that $f(1) = 1$ and $f(x\rhd y) = f(x)\rhd f(y)$ for all $x, y \in X$. Let $u, v \in \mathfrak{x}$ and γ_u (resp. γ_v) be a smooth path in X pointed at 1 such that $\left.\frac{\partial}{\partial s}\right|_{s=0}\gamma_u(s) = u$ (resp. $\left.\frac{\partial}{\partial s}\right|_{s=0}\gamma_v(s) = v$). We have

$$\left.\frac{\partial}{\partial t}\right|_{t=0}(f(\gamma_u(s)\rhd\gamma_v(t))) = T_1 f(Ad_{\gamma_u(s)}(v))$$

and

$$\left.\frac{\partial}{\partial t}\right|_{t=0}(f(\gamma_u(s))\rhd f(\gamma_v(t))) = Ad_{f(\gamma_u(s))}(T_1 f(v)).$$

Now, if we differentiate with respect to the variable s, we have

$$\left.\frac{\partial}{\partial s}\right|_{s=0}(T_1 f(Ad_{\gamma_u(s)}(v))) = T_1 f([u, v])$$

and

$$\left.\frac{\partial}{\partial s}\right|_{s=0}(Ad_{f(\gamma_u(s))}(T_1 f(v))) = [T_1 f(u), T_1 f(v)].$$

That means $T_1 f$ is a morphism of Leibniz algebras.

□

Example 3.1.3 (Group)**.** Let G be a Lie group, then we get in this way the canonical Lie algebra structure on T_1G.

Example 3.1.4 (Augmented rack)**.** Let $X \xrightarrow{p} G$ be an augmented Lie rack, then $T_1X \xrightarrow{T_1p} T_1G$ is a Lie algebra in the category of linear maps (see [LP98]). This structure induces a Leibniz algebra structure on T_1X which is isomorphic to the one induces by the Lie rack structure on X.

We remark that a local smooth structure around 1 is sufficient to provide T_1X with a Leibniz algebra structure.

Proposition 3.1.5. *Let X be a Lie local rack, then T_1X is a Leibniz algebra.*

3.2 From $As_p(X)$-modules to Leibniz representations

Let X be a rack. An $As_p(X)$-module is an abelian group A provided with a morphism of groups $\phi : As_p(X) \to Aut(A)$. By adjointness, this is the same thing as a morphism of pointed racks $\phi : X \to Conj(Aut(A))$.

Definition 3.2.1. *Let X be a Lie rack, a **smooth** $As(X)$**-module** is an $As_p(X)$module A such that*

1. *A is an abelian Lie group.*
2. *$\phi : X \times A \to A$ is smooth.*

Recall that, given a Leibniz algebra $\mathfrak{g}$, a $\mathfrak{g}$-representation $\mathfrak{a}$ is a vector space provided with two linear maps

$$[-,-]_L : \mathfrak{g} \otimes \mathfrak{a} \to \mathfrak{a}$$

and

$$[-,-]_R : \mathfrak{a} \otimes \mathfrak{g} \to \mathfrak{a},$$

satisfying the axioms $(LLM), (LML)$ and (MLL) (cf. Definition 1.2.1).

There are two particular classes of modules. The first, that we called *symmetric*, are the modules where $[-,-]_L = -[-,-]_R$. The second, that we called *anti-symmetric*, are the modules where $[-,-]_R = 0$. In Remark 1.2.5 we have seen that, given a Leibniz algebra $\mathfrak{g}$ and $\mathfrak{a}$ a vector space equipped with a morphism of Leibniz algebra $\phi : \mathfrak{g} \to End(\mathfrak{a})$, we can put two structures of $\mathfrak{g}$-representation on $\mathfrak{a}$. One is *symmetric* and defined by

$$[x,a]_L = \phi_x(a) \text{ and } [a,x]_R = -\phi_x(a), \quad \forall x \in \mathfrak{g}, a \in \mathfrak{a},$$

and the other is *anti-symmetric* and defined by

$$[x,a]_L = \phi_x(a) \text{ and } [a,x]_R = 0, \quad \forall x \in \mathfrak{g}, a \in \mathfrak{a}.$$

Moreover, in Example 2.2.4 and Example 2.2.5, we have seen that, given a rack X and A a (smooth) $As(X)$-module, we can put two structures of (smooth) X-module on A. One is called *symmetric* and defined by

$$\phi_{x,y}(a) = \phi_x(a) \text{ and } \psi_{x,y}(a) = a - \phi_{x \rhd y}(a), \quad \forall x, y \in X, a \in A,$$

and the other is called *anti-symmetric* and defined by

$$\phi_{x,y}(a) = \phi_x(a) \text{ and } \psi_{x,y}(a) = 0, \quad \forall x, y \in X, a \in \mathcal{A}.$$

These constructions are similar to each other because one is the infinitesimal version of the other. Indeed, let (A, ϕ, ψ) be a smooth symmetric X-module. We have by definition two smooth maps

$$\phi : X \times X \times A \to A \text{ and } \psi : X \times X \times A \to A$$

with $\phi_{1,1} = id, \psi_{1,1} = 0$. Thus the differentials of these maps at $(1, 1)$ gives us two maps

$$\epsilon : X \times X \to Aut(\mathfrak{a}); \epsilon(x, y) = T_1\phi_{x,y}$$

and

$$\chi : X \times X \to End(\mathfrak{a}); \chi(x, y) = T_1\psi_{x,y}.$$

These maps are smooth, so we can differentiate them at $(1, 1)$ to obtain

$$T_{(1,1)}\epsilon : \mathfrak{x} \oplus \mathfrak{x} \to End(\mathfrak{a})$$

and

$$T_{(1,1)}\chi : \mathfrak{x} \oplus \mathfrak{x} \to End(\mathfrak{a}).$$

Then we put

$$[-, -]_L : \mathfrak{g} \otimes \mathfrak{a} \to \mathfrak{a}; [u, m]_L = T_{(1,1)}\epsilon(u, 0)(m)$$

and

$$[-, -]_R : \mathfrak{a} \otimes \mathfrak{g} \to \mathfrak{a}; [m, u]_R = T_{(1,1)}\chi(0, u)(m).$$

Lemma 3.2.2. *The linear map $[-, -]_L$ satisfies the axiom (LLM).*

Proof : We want to show that for all $u, v \in \mathfrak{x}, m \in \mathfrak{a}$

$$[u, [v, m]_L]_L = [[u, v], m]_L + [v, [u, m]_L]_L.$$

By hypothesis, ϕ satisfies the following relation for all $x, y, z \in X$

$$\phi_{x,y\rhd z} \circ \phi_{y,z} = \phi_{x\rhd y,x\rhd z} \circ \phi_{x,z},$$

and if we take $z = 1$, we obtain

$$\phi_{x,1} \circ \phi_{y,1} = \phi_{x\rhd y,1} \circ \phi_{x,1},$$

and so

$$T_1\phi_{x,1} \circ T_1\phi_{y,1} = T_1\phi_{x\rhd y,1} \circ T_1\phi_{x,1},$$

that is

$$\epsilon(x, 1) \circ \epsilon(y, 1) = \epsilon(x\rhd y, 1) \circ \epsilon(x, 1).$$

Let $u, v \in \mathfrak{x}$ and γ_u (resp. γ_v) be a smooth path in X pointed at 1 such that $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_u(s) = u$ (resp. $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_v(s) = v$). We have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0} (\epsilon(\gamma_u(s), 1) \circ \epsilon(\gamma_v(t), 1)) &= \epsilon(\gamma_u(s), 1) \circ \left.\frac{\partial}{\partial t}\right|_{t=0} \epsilon(\gamma_v(t), 1) \\ &= \epsilon(\gamma_u(s), 1) \circ [v, -]_L\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial}{\partial s}\right|_{s=0}(\epsilon(\gamma_u(s),1)\circ[v,0]_L) &= (\left.\frac{\partial}{\partial s}\right|_{s=0}\epsilon(\gamma_u(s),1))\circ[v,-]_L\\ &= [u,-]_L\circ[v,-]_L\\ &= [u,[v,-]_L]_L.\end{aligned}$$

On another hand, we have

$$\left.\frac{\partial}{\partial t}\right|_{t=0}(\epsilon(\gamma_u(s)\rhd\gamma_v(t),1)\circ\epsilon(\gamma_u(s),1)) = [Ad_{\gamma_u(s)}(v),-]_L\circ\epsilon(\gamma_u(s),1)$$

and

$$\begin{aligned}\left.\frac{\partial}{\partial s}\right|_{s=0}([Ad_{\gamma_u(s)}(v),-]_L\circ\epsilon(\gamma_u(s),1)) &= \left.\frac{\partial}{\partial s}\right|_{s=0}([Ad_{\gamma_u(s)}(v),-]_L)\circ\epsilon(1,1)\\ &\quad + [Ad_{\gamma_u(0)}(v),-]_L\circ\left.\frac{\partial}{\partial s}\right|_{s=0}(\epsilon(\gamma_u(s),1))\\ &= [[u,v],-]_L+[v,-]_L\circ[u,-]_L\\ &= [[u,v],-]_L+[v,[u,-]]_L.\end{aligned}$$

Hence by identification

$$[u,[v,-]_L]_L = [[u,v],-]_L+[v,[u,-]]_L.$$

That is, $[-,-]_L$ satisfies (LLM).

□

Lemma 3.2.3. *If (A,ϕ,ψ) is symmetric, then $[-,-]_L = -[-,-]_R$ and if (A,ϕ,ψ) is anti-symmetric, then $[-,-]_R = 0$.*

Proof : Suppose that (A,ϕ,ψ) is anti-symmetric, then $\psi = 0$, and it is clear that $[-,-]_R = 0$. Now, suppose that (A,ϕ,ψ) is symmetric, that is $\psi_{x,y}(a) = a - \phi_{x\rhd y,1}(a)$. We have $\chi(x,y) = T_1\psi_{x,y} = id - \epsilon(x\rhd y,1)$. Let $u\in\mathfrak{x}$ and γ_u be a path in X pointed at 1 such that $\left.\frac{\partial}{\partial s}\right|_{s=0}\gamma_u(s) = u$. We have

$$\begin{aligned}[-,u]_R &= T_{(1,1)}\chi(0,u)\\ &= \left.\frac{\partial}{\partial s}\right|_{s=0}\chi(1,\gamma_u(s))\\ &= \left.\frac{\partial}{\partial s}\right|_{s=0}(id-\epsilon(\gamma_u(s),1))\\ &= -\left.\frac{\partial}{\partial s}\right|_{s=0}\epsilon(\gamma_u(s),1)\\ &= -[u,-]_L.\end{aligned}$$

□

Finally, we have shown the following proposition:

Proposition 3.2.4. *Let X be a Lie rack, let $\mathfrak{x}$ be its Leibniz algebra, let A be an abelian Lie group and let $\mathfrak{a}$ be its Lie algebra. If (A,ϕ,ψ) is a smooth symmetric (resp. anti-symmetric) X-module, then $(\mathfrak{a},[-,-]_L,[-,-]_R)$ is a symmetric (resp. anti-symmetric) $\mathfrak{x}$-module.*

3.3 From Lie rack cohomology to Leibniz cohomology

3.3.1 A morphism from Lie rack cohomology to Leibniz cohomology

Proposition 3.3.1. *Let X be a Lie rack and let A be a smooth $As(X)$-module. We have morphisms of cochains complexes*

$$CR^n_p(X,A^s)_s \xrightarrow{\delta^n} CL^n(\mathfrak{x},\mathfrak{a}^s)$$

and

$$CR^n_p(X,A^a)_s \xrightarrow{\delta^n} CL^n(\mathfrak{x},\mathfrak{a}^a),$$

given by $\delta^n(f)(a_1,\ldots,a_n) = d^n f(1,\ldots,1)((a_1,0,\ldots,0),\ldots,(0,\ldots,0,a_n))$ *(where $d^n f$ is the n-th differential of f).*

Proof : Let $f \in CR^n_p(X,A^s)$ and $(x_0,\ldots,x_n) \in X^{n+1}$. We have

$$d_R f(x_0,\ldots,x_n) = \sum_{i=1}^{n+1}(-1)^{i-1}\phi_{x_1\triangleright\ldots\triangleright x_i}(f(x_1,\ldots,\hat{x_i},\ldots,x_{n+1})) - f(x_1,\ldots,x_i\triangleright x_{i+1},\ldots,x_i\triangleright x_{n+1})$$

(Here $\phi_{x_1\triangleright\ldots\triangleright x_i} = \phi_{x_1\triangleright\ldots\triangleright x_i, x_1\triangleright\ldots\triangleright\widehat{x_1}\triangleright\ldots\triangleright x_{n+1}}$ cf. section 3.2)
Let $(\gamma_0(t_0),\ldots,\gamma_n(t_n))$ be a family of paths $\gamma_i :]-\epsilon_i,+\epsilon_i[\to V$ such that

$$\gamma_i(0) = 1 \text{ and } \frac{\partial}{\partial s}_{s=0}\gamma_i(s) = x_i.$$

We want to show that

$$\delta^{n+1}(d^n_R f) = d^n_L(\delta^n(f)).$$

We have by definition

$$\delta^{n+1}(d^n_R f)(a_0,\ldots,a_n) = \left.\frac{\partial^{n+1}}{\partial t_0\ldots\partial t_n}\right|_{t_i=0} df(\gamma_0(t_0),\ldots,\gamma_n(t_n)).$$

Lemma 3.3.2. $\forall i \in \{1,\ldots,n\}$

$$\left.\frac{\partial^{n+1}}{\partial t_0\ldots\partial t_n}\right|_{|t_i=0}\phi_{\gamma_0(t_0)\triangleright\ldots\triangleright\gamma_i(t_i)}(f(\gamma_0(t_0),\ldots,\gamma_{i-1}(t_{i-1}),\gamma_{i+1}(t_{i+1}),\ldots,\gamma_n(t_n))) = a_i.d_n(f)(a_0,\ldots,\widehat{a_i},\ldots,a_n)$$

Proof :
Notation:

$$(t_0,\ldots,t_i)\mapsto\phi_{\gamma_0(t_0)\triangleright\ldots\triangleright\gamma_i(t_i)}$$

will be denoted by

$$(t_0,\ldots,t_i)\mapsto A(t_0,\ldots,t_i)$$

and

$$(t_0,\ldots,\widehat{t_i},\ldots,t_n)\mapsto f(\gamma_0(t_0),\ldots,\gamma_{i-1}(t_{i-1}),\gamma_{i+1}(t_{i+1}),\ldots,\gamma_n(t_n))$$

will be denoted by

$$(t_0,\ldots,\widehat{t_i},\ldots,t_n)\mapsto m(t_0,\ldots,\widehat{t_i},\ldots,t_n)$$

We have

$$
\begin{aligned}
\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_k=0} A(t_0,\dots,t_i)(m(t_0,\dots,t_n)) &= \frac{\partial^{n}}{\partial t_0 \dots \partial t_{n-1}}\Big|_{t_k=0} \Big(\frac{\partial}{\partial t_n}\Big|_{t_n=0} A(t_0,\dots,t_i)(m(t_0,\dots,0))\\
&+ \frac{\partial^{n}}{\partial t_0 \dots \partial t_{n-1}}\Big|_{t_k=0} \Big(A(t_0,\dots,t_i)\Big(\frac{\partial}{\partial t_n}\Big|_{t_n=0} m(t_0,\dots,t_n)\Big)\Big)\\
&= \frac{\partial^{n}}{\partial t_0 \dots \partial t_{n-1}}\Big|_{t_k=0} \Big(A(t_0,\dots,t_i)\Big(\frac{\partial}{\partial t_n}\Big|_{t_n=0} m(t_0,\dots,t_n)\Big)\Big)
\end{aligned}
$$

because $f(x_0,\dots,1,\dots,x_n)=0$ (cf. Definition 2.3.11).
So by the same argument we have

$$
\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_k=0} A(t_0,\dots,t_i)(m(t_0,\dots,t_n)) = \frac{\partial}{\partial t_i}\Big|_{t_i=0} A(0,\dots,0,t_i)\Big(\frac{\partial^{n}}{\partial t_0 \dots \partial t_n}\Big|_{t_k=0} m(t_0,\dots,t_n)\Big)
$$

Hence

$$
\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_i=0} A(t_0,\dots,t_n)(m(t_0,\dots,t_n)) = x_i.d_n(f)(x_0,\dots,\hat{x_i},\dots,x_n)
$$

□

Lemma 3.3.3. $\forall i \in \{1,\dots,n\}$

$$
\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_i=0} f(\gamma_0(t_0),\dots,\gamma_i(t_i)\triangleright\gamma_{i+1}(t_{i+1}),\dots,\gamma_i(t_i)\triangleright\gamma_n(t_n)) = \sum_{k=i+1}^{n} \delta_n f(a_0,\dots,[a_i,a_k],\dots,a_n)
$$

Proof : We have $\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_i=0} f(\gamma_0(t_0),\dots,\gamma_i(t_i)\triangleright\gamma_{i+1}(t_{i+1}),\dots,\gamma_i(t_i)\triangleright\gamma_n(t_n))$ which is equal to

$$
\frac{\partial}{\partial t_i}\Big|_{t_i=0} d^n f(1,\dots,1)((a_0,0,\dots,0),\dots,(0,\dots,Ad_{\gamma_i(t_i)}(a_{i+1}),\dots,0),\dots,(0,\dots,0,Ad_{\gamma_i(t_i)}(a_n))
$$

Moreover, this is equal to

$$
\sum_{k=i+1}^{n} d^n f(1,\dots,1)((a_0,0,\dots,0),\dots,(0,\dots,[a_i,a_k],\dots,0),\dots,(0,\dots,0,a_n))
$$

and this expression is equal to $\sum_{k=i+1}^{n} \delta^n f(a_0,\dots,[a_i,a_k],\dots,a_n)$.

□

So

$$
\begin{aligned}
\delta^{n+1}(d_R^n f)(a_0,\dots,a_n) &= \sum_{i=0}^{n} (-1)^i \Big(a_i.\delta^n(f)(a_0,\dots,\hat{a_i},\dots,a_n) - \sum_{k=i+1}^{n} \delta^n f(a_0,\dots,[a_i,a_k],\dots,a_n)\Big)\\
&= \sum_{i=0}^{n} (-1)^i a_i.\delta^n(f)(a_0,\dots,\hat{a_i},\dots,a_n) + \sum_{0\le i<k\le n} (-1)^{i+1}\delta^n f(a_0,\dots,[a_i,a_k],\dots,a_n)
\end{aligned}
$$

that is

$$
\delta^{n+1}(d_R^n f) = d_L^n(\delta^n(f))
$$

This is exactly the same proof as for the case where A is anti-symmetric.

□

We remark that we only need a local cocyle identity around 1. Thus we have

Proposition 3.3.4. *Let X be a Lie rack, let U be a 1-neighborhood in X and let A be a smooth $As(X)$-module. We have morphisms of cochain complexes*

$$CR_p^n(U, A^s) \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^s)$$

and

$$CR_p^n(U, A^a) \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^a),$$

given by $\delta^n(f)(a_0, \ldots, a_n) = d^n f(1, \ldots, 1)((a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_n))$.

3.3.2 Induced morphism from Lie cohomology to Leibniz cohomology

Let G be a Lie group and A a smooth G-module. We have stated in Proposition 2.3.24 that there are maps $H^n(G, A) \xrightarrow{\Delta^n} HR^n(G, A^s)$ for all $n \in \mathbb{N}$. These maps still exist when the cocycles are smooth, that is we have maps $H^n(G, A)_s \xrightarrow{\Delta^n} HR^n(G, A^s)_s$ for all $n \in \mathbb{N}$. On the other hand, we have shown that for $\mathfrak{g}$ a Lie algebra and $\mathfrak{a}$ a $\mathfrak{g}$-module, there are maps $H^n(\mathfrak{g}, \mathfrak{a}) \xrightarrow{i_n} HL^n(\mathfrak{g}, \mathfrak{a}^s)$ for all $n \in \mathbb{N}$. K.H. Neeb defines in [Nee04], maps $H^n(G, A)_s \xrightarrow{D^n} H^n(\mathfrak{g}, \mathfrak{a})$ for all $n \in \mathbb{N}$, and we have defined maps $HR^n(G, A^s) \xrightarrow{\delta^n} HL^n(\mathfrak{g}, \mathfrak{a}^s)$ for all $n \in \mathbb{N}$. It is natural to believe that there is a link between those maps. The Proposition 3.3.6 confirms this belief.

First of all, recall the maps $\{D^n\}_{n\in\mathbb{N}}$ defined in [Nee04] (Theorem B.6).

Proposition 3.3.5 (Van Est)**.** *Let G be a Lie group and let A be a smooth G-module. We have morphisms of cochain complexes*

$$C_s^n(G, A) \xrightarrow{D^n} C^n(\mathfrak{g}, \mathfrak{a})$$

given by $D^n(f)(a_1, \ldots, a_n) = \sum_{\sigma\in\sigma_n} (-1)^{sign(\sigma)} d^n f(1, \ldots, 1)((a_{\sigma(1)}, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_{\sigma(n)}))$.

Proof : It is the content of Theorem B.6 in [Nee04].

□

Proposition 3.3.6. *Let G be a Lie group and let A be a smooth G-module. The diagram*

$$\begin{tikzcd}
\{C^n(G, A)_s, d^n\}_{n\in\mathbb{N}} \arrow[r, "\{\Delta^n\}_{n\in\mathbb{N}}"] \arrow[d, "\{D^n\}_{n\in\mathbb{N}}"'] & \{CR_p^n(G, A^s)_s, d_R^n\}_{n\in\mathbb{N}} \arrow[d, "\{\delta^n\}_{n\in\mathbb{N}}"] \\
\{C^n(\mathfrak{g}, \mathfrak{a}), d^n\}_{n\in\mathbb{N}} \arrow[r, "\{i_n\}_{n\in\mathbb{N}}"'] & \{CL^n(\mathfrak{g}, \mathfrak{a}^s), d_L^n\}_{n\in\mathbb{N}}
\end{tikzcd}$$

is a commutative diagram of morphisms of cochain complexes.

To prove this result, we need the following lemma.

Lemma 3.3.7. *Let $1 \leq k \leq n$, $f \in C^n(G,A)_s$, $(g_1,\dots,g_n) \in G^n$ and $(v_1,\dots,v_k) \in (\mathfrak{g}^n)^k$ such that*

$$g_{i_1} = \cdots = g_{i_{k+1}} = 1$$

for $1 \leq i_1 < \cdots < i_{k+1} \leq n$, and

$$v_j = (0,\dots,0,x_{i_j},0,\dots,0)$$

for $x_{i_j} \in \mathfrak{g}$.
We have

$$d^k f(g_1,\dots,g_n)(v_1,\dots,v_k) = 0.$$

Proof : Let γ_{i_j} be a path in G such that $\gamma_{i_j}(0) = 1$ and $\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_{i_j}(t) = x_{i_j}$. We have $f \in C^n(G,A)_s$, thus

$$f(g_1,\dots,\gamma_{i_1}(t_1),\dots,\gamma_{i_k}(t_k),\dots,1,\dots,g_n) = 0.$$

Moreover, we have

$$\left.\frac{\partial^k}{\partial t_1 \dots \partial t_k}\right|_{t_1\dots t_k=0} f(g_1,\dots,\gamma_{i_1}(t_1),\dots,\gamma_{i_k}(t_k),\dots,1,\dots,g_n) = d^k f(g_1,\dots,g_n)(v_1,\dots,v_k).$$

Hence,

$$d^k f(g_1,\dots,g_n)(v_1,\dots,v_k) = 0.$$

$\square$

Proof of the Proposition : We have to show that for all $n \in \mathbb{N}$,

$$\delta^n \circ \Delta^n = i^n \circ D^n.$$

Let $f \in C^n(G,A)$, $x_1,\dots,x_n \in \mathfrak{g}$ and $\gamma_1,\dots,\gamma_n$ paths in G, such that $\gamma_i(0) = 1$ and $\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_i(t) = x_i$. We have

$$\begin{aligned}
\delta^n(\Delta^n(f))(x_1,\dots,x_n) &= \left.\frac{\partial^n}{\partial t_1 \dots \partial t_n}\right|_{t_1\dots t_n=0} \Delta^n(f)(\gamma_1(t_1),\dots,\gamma_n(t_n)) \\
&= \sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)} \left.\frac{\partial^n}{\partial t_1 \dots \partial t_n}\right|_{t_1\dots t_n=0} f(\gamma_1^\sigma(t_1,\dots,t_n),\dots,\gamma_n^\sigma(t_1,\dots,t_n))
\end{aligned}$$

where

$$\gamma_k^\sigma(t_1,\dots,t_n) = \gamma_{j_1}(t_{j_1}) \rhd \dots \rhd \gamma_{j_l}(t_{j_l}) \rhd \gamma_{\sigma(k)}(t_{\sigma(k)})$$

with $j_1 < \cdots < j_l < \sigma(k)$ and $j_i \notin \sigma(1),\dots,\sigma(k)$.

Let $1 \leq k \leq n$, suppose that the following formula is true

$$\left.\frac{\partial^{n-k+1}}{\partial t_k \dots \partial t_n}\right|_{t_k\dots t_n=0} f(\gamma_1^\sigma(t_1,\dots,t_n),\dots,\gamma_n^\sigma(t_1,\dots,t_n)) = d^{n-k+1} f(g_1,\dots,g_n)(v_1,\dots,v_{n-k+1})$$

where

$$g_i = \begin{cases} 1 & \text{if } i \in \{\sigma^{-1}(k),\dots,\sigma^{-1}(n)\} \\ \gamma_i^\sigma(t_1,\dots,t_k) & \text{else if} \end{cases}$$

and $v_i = (v_i^1, \ldots, v_i^n)$ with

$$v_i^j = \begin{cases} 0 & \text{if } j \neq \sigma^{-1}(k-1+i) \\ \left.\dfrac{\partial}{\partial t_j}\right|_{t_j=0} \gamma^\sigma_{\sigma^{-1}(j)}(t_1, \ldots, t_{k-1}, 1, \ldots, 1, t_j) & \text{else if} \end{cases}$$

We have

$$\left.\frac{\partial^{n-k+2}}{\partial t_{k-1} \ldots \partial t_n}\right|_{t_{k-1} \ldots t_n = 0} f(\gamma_1^\sigma(t_1, \ldots, t_n), \ldots, \gamma_n^\sigma(t_1, \ldots, t_n)) = \left.\frac{\partial}{\partial t_{k-1}}\right|_{t_{k-1}=0} (d^{n-k+1} f(g_1, \ldots, g_n)(v_1, \ldots, v_{n-k+1}))$$

and

$$\left.\frac{\partial}{\partial t_{k-1}}\right|_{t_{k-1}=0} (d^{n-k+1} f(g_1, \ldots, g_n)(v_1, \ldots, v_{n-k+1})) = d^{n-k+2} f(h_1, \ldots, h_n)(w_1, \ldots, w_{n-k+2})$$
$$+ \sum_{i=1}^{k} d^{n-k+1} f(h_1, \ldots, h_n)(v_1(t_1, \ldots, t_{k-2}, 1), \ldots, \left.\frac{\partial}{\partial t_{k-1}}\right|_{t_{k-1}=0} v_i(t_1, \ldots, t_{k-1}), \ldots, v_k(t_1, \ldots, t_{k-2}, 1))$$

where

$$h_i = \begin{cases} 1 & \text{if } i \in \{\sigma^{-1}(k-1), \ldots, \sigma^{-1}(n)\} \\ \gamma_i^\sigma(t_1, \ldots, t_{k-1}) & \text{else if} \end{cases}$$

and $w_1 = (w_1^1, \ldots, w_1^n)$ with

$$w_1^j = \begin{cases} 0 & \text{if } j \neq \sigma^{-1}(k-1) \\ \left.\dfrac{\partial}{\partial t_{k-1}}\right|_{t_{k-1}=0} \gamma^\sigma_{\sigma^{-1}(k-1)}(t_1, \ldots, t_{k-1}) & \text{else if} \end{cases}$$

and $w_i^j(t_1, \ldots, t_{k-2}) = v_{i-1}^j(t_1, \ldots, t_{k-2}, 1)$.

By Lemma 3.3.7, the term

$$\sum_{i=1}^{k} d^{n-k+1} f(h_1, \ldots, h_n)(v_1(t_1, \ldots, t_{k-2}, 1), \ldots, \left.\frac{\partial}{\partial t_{k-1}}\right|_{t_{k-1}=0} v_i(t_1, \ldots, t_{k-1}), \ldots, v_k(t_1, \ldots, t_{k-2}, 1))$$

is equal to zero. Thus, if the formula (3.3.2) is true, then we have

$$\left.\frac{\partial^{n-k+2}}{\partial t_{k-1} \ldots \partial t_n}\right|_{t_{k-1} \ldots t_n = 0} f(\gamma_1^\sigma(t_1, \ldots, t_n), \ldots, \gamma_n^\sigma(t_1, \ldots, t_n)) = d^{n-k+2} f(g_1, \ldots, g_n)(v_1, \ldots, v_{n-k+1}).$$

As this formula is true for $k = n$, by induction this formula is true for $k = 1$. That is,

$$\left.\frac{\partial^n}{\partial t_1 \ldots \partial t_n}\right|_{t_1 \ldots t_n = 0} f(\gamma_1^\sigma(t_1, \ldots, t_n), \ldots, \gamma_n^\sigma(t_1, \ldots, t_n)) = d^n f(g_1, \ldots, g_n)(v_1, \ldots, v_n).$$

Hence,

$$\begin{aligned} \delta^n(\Delta^n(f))(x_1, \ldots, x_n) &= \sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)} d^n f(1, \ldots, 1)(v_1, \ldots, v_n) \\ &= i^n(D^n(f))(x_1, \ldots, x_n). \end{aligned}$$

Thus

$$\delta^n \circ \Delta^n = i^n \circ D^n.$$

□

Hence we have for all $n \in \mathbb{N}$ a commutative diagram

$$\begin{tikzcd} H^n(G,A)_s \arrow[r, "{[\Delta^n]}"] \arrow[d, "{[D^n]}"'] & HR^n_s(G,A^s) \arrow[d, "{[\delta^n]}"] \\ H^n(\mathfrak{g},\mathfrak{a}) \arrow[r, "{[i_n]}"'] & HL^n(\mathfrak{g},\mathfrak{a}^s) \end{tikzcd}$$

In [Nee04], K.H. Neeb shows in particular that, if the Lie group G is simply connected, then D^2 is an isomorphism and its inverse is given by

$$\iota^2(\omega)(g,h) = \int_{\gamma_{g,h}} \omega^{eq}$$

where $\gamma_{g,h}$ is a smooth singular 2-chain in G such that $\partial\gamma_{g,h} = \gamma_g - \gamma_{gh} + g\gamma_h$ and γ_g (resp. γ_h) a smooth path in G from 1 to g (resp. h).

3.4 From Leibniz cohomology to Lie local rack cohomology

In this section, we study two cases of Leibniz cocyles integration. This section will be used in the following section to integrate a Leibniz algebra into a local augmented Lie rack.
First, we study the integration of a 1-cocycle in $ZL^1(\mathfrak{g},\mathfrak{a}^s)$ into a Lie rack 1-cocycle in $ZR^1_p(G,\mathfrak{a}^s)_s$, where G is a simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{a}$ a representation of G.
Secondly, we use the result of the first part to study the integration of a 2-cocycle in $ZL^2(\mathfrak{g},\mathfrak{a}^a)$ into a local Lie rack 2-cocycle in $ZR^2_p(U,\mathfrak{a}^a)_s$, where U is a 1-neighborhood in a simply connected Lie group G with Lie algebra $\mathfrak{g}$, and $\mathfrak{a}$ a representation of G. It is this second part that we will use to integrate Leibniz algebras.

3.4.1 From Leibniz 1-cocycles to Lie rack 1-cocycles

Let G be a simply connected Lie group and $\mathfrak{a}$ a representation of G. We want to define a morphism I^1 from $ZL^1(\mathfrak{g},\mathfrak{a}^s)$ to $ZR^1_p(G,\mathfrak{a}^s)_s$ which sends $BL^1(\mathfrak{g},\mathfrak{a}^s)$ into $BR^1_p(G,\mathfrak{a}^s)_s$. For this, we put

$$I^1(\omega)(g) = \int_{\gamma_g} \omega^{eq},$$

where $\omega \in ZL^1(\mathfrak{g},\mathfrak{a}^s)$, $\gamma : G \times [0,1] \to G$ is a smooth map such that γ_g is a path from 1 to g, γ_1 is the constant path equal to 1, and ω^{eq} is the closed left equivariant differential form in $\Omega^1(G,\mathfrak{a})$ defined by

$$\omega^{eq}(g)(m) = g.(\omega(T_gL_{g^{-1}}(m))).$$

By definition, it is clear that $I^1(\omega)(1) = 0$.

For the moment, $I^1(\omega)$ depends on γ, but because ω is a cocycle, the dependance with respect to γ disappears.

Proposition 3.4.1. *I^1 does not depend on γ.*

Proof : Let $\gamma, \gamma' : G \times [0,1] \to G$ such that $\gamma_g(0) = \gamma'_g(0) = 1$ and $\gamma_g(1) = \gamma'_g(1) = g$. We are going to show that

$$\int_{\gamma_g} \omega^{eq} = \int_{\gamma'_g} \omega^{eq}.$$

We have $\int_{\gamma_g} \omega^{eq} - \int_{\gamma'_g} \omega^{eq} = \int_{\gamma_g - \gamma'_g} \omega^{eq}$. As $H_1(G) = 0$ and $\partial(\gamma_g - \gamma'_g) = 0$, there exists $\sigma_g : [0,1]^2 \to G$ such that $\gamma_g - \gamma'_g = \partial\sigma_g$. So

$$\begin{aligned}
\int_{\gamma_g} \omega^{eq} - \int_{\gamma'_g} \omega^{eq} &= \int_{\gamma_g - \gamma'_g} \omega^{eq} \\
&= \int_{\partial\sigma_g} \omega^{eq} \\
&= \int_{\sigma_g} d_{dR}\omega^{eq} \\
&= 0.
\end{aligned}$$

Hence I^1 does not depend on γ.

□

Proposition 3.4.2. *I^1 sends cocycles to cocycles and coboundaries to coboundaries.*

Proof : First, let $\omega \in ZL^1(\mathfrak{g}, \mathfrak{a}^s)$, we have

$$\begin{aligned}
d_R I(\omega)(g,h) &= g.I(\omega)(h) - I(\omega)(g \triangleright h) - (g \triangleright h).I(\omega)(g) + I(\omega)(g) \\
&= g.\int_{\gamma_h} \omega^{eq} - \int_{\gamma_{g\triangleright h}} \omega^{eq} - (g\triangleright h).\int_{\gamma_g} \omega^{eq} + \int_{\gamma_g} \omega^{eq} \\
&= \int_{\gamma_h} g.\omega^{eq} - \int_{\gamma_{g\triangleright h}} \omega^{eq} - \int_{\gamma_g} (g\triangleright h).\omega^{eq} + \int_{\gamma_g} \omega^{eq} \\
&= \int_{g\gamma_h} \omega^{eq} - \int_{\gamma_{g\triangleright h}} \omega^{eq} - \int_{(g\triangleright h)\gamma_g} \omega^{eq} + \int_{\gamma_g} \omega^{eq} \\
&= \int_{g\gamma_h - \gamma_{g\triangleright h} - (g\triangleright h)\gamma_g + \gamma_g} \omega^{eq}.
\end{aligned}$$

As $H^1(G) = 0$ and $\partial(g\gamma_h - \gamma_{g\triangleright h} - (g\triangleright h)\gamma_g + \gamma_g) = 0$, there exists $\gamma_{g,h} : [0,1]^2 \to G$ such that $\partial\gamma_{g,h} = g\gamma_h - \gamma_{g\triangleright h} - (g\triangleright h)\gamma_g + \gamma_g$. Hence, we have

$$\begin{aligned}
d_R I(\omega)(g,h) &= \int_{\partial\gamma_{g,h}} \omega^{eq} \\
&= \int_{\gamma_{g,h}} d_{dR}\omega^{eq} \\
&= 0.
\end{aligned}$$

Hence $ZL^1(\mathfrak{g}, \mathfrak{a}^s)$ is sent to $ZR^1_p(G, \mathfrak{a}^s)_s$.

Secondly, let $\omega \in BL^1(\mathfrak{g}, \mathfrak{a}^s)$. There exists $\beta \in \mathfrak{a}$ such that $\omega(m) = m.\beta$. We have

$$\begin{aligned} I(\omega)(g) &= \int_{\gamma_g} \omega^{eq} \\ &= \int_{\gamma_g} (d_L\beta)^{eq} \\ &= \int_{\gamma_g} d_{dR}\beta^{eq} \\ &= \beta^{eq}(g) - \beta^{eq}(1) \\ &= g.\beta - \beta \\ &= d_R\beta(g). \end{aligned}$$

Hence $BL^1(\mathfrak{g}, \mathfrak{a}^s)$ is sent to $BR^1_p(G, \mathfrak{a}^s)_s$.

□

Proposition 3.4.3. *I^1 is a left inverse for δ^1.*

Proof : Let $\omega \in ZL^1(\mathfrak{g}, \mathfrak{a}^s)$. Let $\varphi : U \to \mathfrak{g}$ be a local chart around 1 such that $\varphi(1) = 0$ and $d\varphi^{-1}(0) = id$.
We define for $x \in \mathfrak{g}$ the smooth map $\alpha_x :]-\epsilon, +\epsilon[\to U$ by setting

$$\alpha_x(s) = \varphi^{-1}(sx),$$

and we define for all $s \in]-\epsilon, +\epsilon[$, the smooth map $\gamma_{\alpha_x(s)} : [0,1] \to U$ by setting

$$\gamma_{\alpha_x(s)}(t) = \varphi^{-1}(tsx).$$

We have

$$\begin{aligned} \delta^1(I^1(\omega))(x) &= \frac{\partial}{\partial s}\Big|_{s=0} I^1(\omega)(\alpha_x(s)) \\ &= \frac{\partial}{\partial s}\Big|_{s=0} \int_{\gamma_{\alpha_x(s)}} \omega^{eq} \\ &= \frac{\partial}{\partial s}\Big|_{s=0} \int_{[0,1]} \gamma^*_{\alpha_x(s)}\omega^{eq} \\ &= \frac{\partial}{\partial s}\Big|_{s=0} \int_{[0,1]} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_{\alpha_x(s)}(t))dt. \end{aligned}$$

Moreover, $\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma_{\alpha_x(s)}(t) = \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi^{-1}(stx) = sx$, thus

$$\begin{aligned}
\delta^1(I^1(\omega))(x) &= \left.\frac{\partial}{\partial s}\right|_{s=0} \int_{[0,1]} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(sx)dt \\
&= \int_{[0,1]} \left.\frac{\partial}{\partial s}\right|_{s=0} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(sx)dt \\
&= \int_{[0,1]} \left.\frac{\partial}{\partial s}\right|_{s=0} \omega^{eq}(\varphi^{-1}(tsx))(sx)dt \\
&= \int_{[0,1]} \left.\frac{\partial}{\partial s}\right|_{s=0} (\varphi^{-1})^*\omega^{eq}(tsx)(sx)dt \\
&= \int_{[0,1]} \left.\frac{\partial}{\partial s}\right|_{s=0} s(\varphi^{-1})^*\omega^{eq}(tsx)(x)dt \\
&= \int_{[0,1]} (\varphi^{-1})^*\omega^{eq}(0)(x)dt \\
&= \omega(x) \int_{[0,1]} dt \\
&= \omega(x).
\end{aligned}$$

Hence $\delta^1 \circ I^1 = id$.

□

Remark 3.4.4. The morphism $H^1(\mathfrak{g}, \mathfrak{a}) \overset{D^1}{\to} H^1_s(G, \mathfrak{a})$ is an isomorphism (cf. for example [Nee04]). We can remark that I^1 is the same map as the one defined in [Nee04] to prove the surjectivity. In fact, to prove the Proposition 3.4.2, we just compose the map D^1 and the map Δ^1 defined in Proposition 2.3.24 (Precisely, we use the remark below this proposition which states that $H^1_s(G, \mathfrak{a})_s \overset{\Delta^1}{\to} HR^1_s(G, \mathfrak{a}^s)$ is an injection).

3.4.2 From Leibniz 2-cocycles to Lie local rack 2-cocycles

Let G be a simply connected Lie group, let U be a 1-neigbourhood in G such that log is defined on U and let $\mathfrak{a}$ be a representation of G. In Proposition 3.3.4 we have defined for all $n \in \mathbb{N}$ the maps

$$HR^n_s(U, \mathfrak{a}^a) \overset{[\delta^n]}{\longrightarrow} HL^n(\mathfrak{g}, \mathfrak{a}^a).$$

In the next section, we will see that a Leibniz algebra can be integrated into a local Lie rack since the morphism $[\delta^2]$ is surjective. More precisely, if we can construct a left inverse for $[\delta^2]$, then it gives us an explicit method to construct the local Lie rack which integrates the Leibniz algebra.

In this section, we define a morphism $[I^2]$ from $HL^2(\mathfrak{g}, \mathfrak{a}^a)$ to $HR^2_s(U, \mathfrak{a}^a)$, and we show that it is a left inverse for $[\delta^2]$. To construct the map $[I^2]$, we adapt an integration method of Lie algebra cocycles into Lie group cocycles by integration over simplex. This method is due to W.T. Van Est ([Est54]) and used by K.H. Neeb ([Nee02, Nee04]) for the infinite dimensional case.

Definition of I^2

We want to define a map from $ZL^2(\mathfrak{g},\mathfrak{a}^a)$ to $ZR^2_p(U,\mathfrak{a}^a)_s$ such that $BL^2(\mathfrak{g},\mathfrak{a}^a)$ is sent to $BR^2_p(U,\mathfrak{a}^a)_s$. In the previous section, we have integrated a Leibniz 1-cocycle on a Lie algebra $\mathfrak{g}$ with coefficients in a symmetric module $\mathfrak{a}^s$. In Proposition 1.3.16, we have shown that there is an isomorphism between $CL^2(\mathfrak{g},\mathfrak{a}^a)$ and $CL^1(\mathfrak{g},Hom(\mathfrak{g},\mathfrak{a})^s)$, which sends $ZL^2(\mathfrak{g},\mathfrak{a}^a)$ to $ZL^1(\mathfrak{g},Hom(\mathfrak{g},\mathfrak{a})^s)$ and $BL^2(\mathfrak{g},\mathfrak{a}^a)$ to $BL^1(\mathfrak{g},Hom(\mathfrak{g},\mathfrak{a})^s)$. Hence, we can define a map

$$I : ZL^2(\mathfrak{g},\mathfrak{a}^a) \to ZR^1_p(G,Hom(\mathfrak{g},\mathfrak{a})^s)_s,$$

which sends $BL^2(\mathfrak{g},\mathfrak{a}^a)$ into $BR^1_p(G,Hom(\mathfrak{g},\mathfrak{a})^s)_s$. This is the composition

$$ZL^2(\mathfrak{g},\mathfrak{a}^a) \xrightarrow{\tau^2} ZL^1(\mathfrak{g},Hom(\mathfrak{g},\mathfrak{a})^s) \xrightarrow{I^1} ZR^1_p(G,Hom(\mathfrak{g},\mathfrak{a})^s)_s.$$

Now, we want to define a map from $ZR^1_p(G,Hom(\mathfrak{g},\mathfrak{a})^s)_s$ to $ZR^2_p(U,\mathfrak{a}^a)$. Let $\beta \in CR^1_p(G,Hom(\mathfrak{g},\mathfrak{a})^s)_s$, β has values in the representation $Hom(\mathfrak{g},\mathfrak{a})$, so for all $g \in G$, we can consider the equivariant differential form $\beta(g)^{eq} \in \Omega^1(G,\mathfrak{a})$ defined by

$$\beta(g)^{eq}(h)(m) := h.(\beta(g)(T_hL_{h^{-1}}(m))).$$

Then we define an element in $CR^2_p(U,\mathfrak{a}^a)$ by setting

$$f(g,h) = \int_{\gamma_{g\rhd h}} (\beta(g))^{eq},$$

where $\gamma : G\times[0,1] \to G$ is a smooth map such that for all $g \in G$, γ_g is a path from 1 to g in G and $\gamma_1 = 1$.

For the moment, an element of $ZR^1_p(G,Hom(\mathfrak{g},\mathfrak{a})^s)_s$ is not necessarily sent to an element of $ZR^2_p(U,\mathfrak{a}^a)_s$. To reach our objective, we have to specify the map γ, and we define it by setting

$$\gamma_g(s) = exp(s\log(g)).$$

Then, we define $I^2 : ZL^2(\mathfrak{g},\mathfrak{a}^a) \to CR^2_p(U,\mathfrak{a}^a)_s$ by setting for all $(g,h) \in U_{2-loc}$

$$I^2(\omega)(g,h) = \int_{\gamma_{g\rhd h}} (I(\omega)(g))^{eq}.$$

By definition, it is clear that $I^2(\omega)(g,1) = I^2(\omega)(1,g) = 0$.

Properties of I^2

Proposition 3.4.5. *I^2 sends $ZL^2(\mathfrak{g},\mathfrak{a}^a)$ into $ZR^2_p(U,\mathfrak{a}^a)_s$.*

To prove this proposition we need some lemmas.

Lemma 3.4.6. *For all $(g,h) \in U_{2-loc}$, we have $\gamma_{g\rhd h} = g\rhd\gamma_h$.*

Proof : Let $(g,h) \in U_{2-loc}$, by definition we have

$$\gamma_{g\rhd h}(s) = exp(s(\log(g\rhd h))).$$

By naturality of the exponential, and a fortiori of the logarithm, we have

$$\begin{aligned} exp(s(log(g\rhd h))) &= exp(sAd_g(\log(h))) \\ &= exp(Ad_g(s\log(h))) \\ &= c_g(exp(s\log(h))) \\ &= c_g(\gamma_h(s)). \end{aligned}$$

Hence $\gamma_{g\rhd h} = g\rhd\gamma_h$.

$\square$

Lemma 3.4.7. *Let G be a Lie group, let $\mathfrak{a}$ be a representation of G and $\omega \in Hom(\mathfrak{g}, \mathfrak{a})$. We have for all $g \in G$*

$$g.(\omega^{eq}) = c_g^*((g.\omega)^{eq}).$$

Proof : Let $g, h \in G$ and $x \in T_h G$, we have:

$$\begin{aligned}
(c_g^*((g.\omega)^{eq}))(h)(x) &= (g.\omega)^{eq}(g \rhd h)(d_h c_g(x)) \\
&= (g \rhd h).(g.\omega)(d_{g \rhd h} L_{g \rhd h^{-1}}(d_h c_g(x))) \\
&= (g \rhd h).(g.\omega)(d_h(c_g \circ L_{h^{-1}})(x)) \\
&= (g \rhd h).(g.(\omega(Ad(g^{-1})(d_h(c_g \circ L_{h^{-1}})(x))))) \\
&= gh.(\omega(d_h(c_{g^{-1}} \circ c_g \circ L_{h^{-1}})(x))) \\
&= g.(h.(\omega(d_h L_{h^{-1}}(x)))) \\
&= g.(\omega^{eq}(h)(x)).
\end{aligned}$$

Hence $c_g^*((g.\omega)^{eq}) = g.(\omega)^{eq}$.

$\square$

Proof of proposition : Let $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ and $(g, h, k) \in U_{3-loc}$. We have

$$\begin{aligned}
d_R(I^2(\omega))(g,h,k) &= g.I^2(\omega)(h,k) - I^2(\omega)(g \rhd h, g \rhd k) - (g \rhd h).I^2(\omega)(g,k) + I^2(\omega)(g, h \rhd k) \\
&= g.\int_{\gamma_{h \rhd k}} (I(\omega)(h))^{eq} - \int_{\gamma_{g \rhd (h \rhd k)}} (I(\omega)(g \rhd h))^{eq} - (g \rhd h).\int_{\gamma_{g \rhd k}} (I(\omega)(g))^{eq} \\
&+ \int_{\gamma_{g \rhd (h \rhd k)}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{h \rhd k}} g.((I(\omega)(h))^{eq}) - \int_{\gamma_{g \rhd (h \rhd k)}} (I(\omega)(g \rhd h))^{eq} - \int_{\gamma_{g \rhd k}} (g \rhd h).((I(\omega)(g))^{eq}) \\
&+ \int_{\gamma_{g \rhd (h \rhd k)}} (I(\omega)(g))^{eq}.
\end{aligned}$$

Because of Lemma 3.4.7, we have

$$\begin{aligned}
d_R(I^2(\omega))(g,h,k) &= \int_{\gamma_{h \rhd k}} c_g^*((g.I(\omega)(h))^{eq}) - \int_{\gamma_{g \rhd (h \rhd k)}} I(\omega)(g \rhd h)^{eq} - \int_{\gamma_{g \rhd k}} c_{g \rhd h}^*(((g \rhd h).I(\omega)(g))^{eq}) \\
&+ \int_{\gamma_{g \rhd (h \rhd k)}} I(\omega)(g)^{eq} \\
&= \int_{c_g \circ \gamma_{h \rhd k}} (g.I(\omega)(h)^{eq} - \int_{\gamma_{g \rhd (h \rhd k)}} I(\omega)(g \rhd h)^{eq} - \int_{c_{g \rhd h} \circ \gamma_{g \rhd k}} ((g \rhd h).I(\omega)(g))^{eq} \\
&+ \int_{\gamma_{g \rhd (h \rhd k)}} I(\omega)(g)^{eq},
\end{aligned}$$

and because of Lemma 3.4.6, we have

$$\begin{aligned} d_R(I^2(\omega))(g,h,k) &= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g\rhd(h\rhd k)}} I(\omega)(g\rhd h)^{eq} - \int_{\gamma_{g\rhd(h\rhd k}} ((g\rhd h).I(\omega)(g))^{eq} \\ &+ \int_{\gamma_{g\rhd(h\rhd k)}} I(\omega)(g)^{eq} \\ &= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h))^{eq} - I(\omega)(g\rhd h)^{eq} - ((g\rhd h).I(\omega)(g))^{eq} + I(\omega)(g)^{eq} \\ &= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h) - I(\omega((g\rhd h) - (g\rhd h).I(\omega)(g) + I(\omega)(g))^{eq} \\ &= \int_{\gamma_{g\rhd(h\rhd k)}} d_R(I(\omega))(g,h) \\ &= 0. \end{aligned}$$

Hence $ZL^2(\mathfrak{g},\mathfrak{a}^a)$ is sent to $ZR^2_p(U,\mathfrak{a}^a)_s$.

□

Proposition 3.4.8. *I^2 sends $BL^2(\mathfrak{g},\mathfrak{a}^a)$ into $BR^2_p(U,\mathfrak{a}^a)_s$.*

Proof : Let $\omega \in BL^2(\mathfrak{g},\mathfrak{a}^a)$, there exists an element $\beta \in CL^1(\mathfrak{g},\mathfrak{a}^a)$ such that $\omega = d_L\beta$. We have

$$\begin{aligned} I(\omega)(g) &= I^1(\tau^2(\omega))(g) \\ &= \int_{\gamma_g} (\tau^2(\omega))^{eq} \\ &= \int_{\gamma_g} (\tau^2(d_L\beta))^{eq}. \end{aligned}$$

The fact that $\{\tau^n\}_{n\in\mathbb{N}}$ is a morphism of cochain complexes implies that

$$\begin{aligned} I(\omega)(g) &= \int_{\gamma_g} (d(\tau^1(\beta)))^{eq} \\ &= \int_{\gamma_g} d_{dR}((\tau^1(\beta))^{eq}) \\ &= \int_{\partial\gamma_g} (\tau^1(\beta))^{eq} \\ &= g.\beta - \beta. \end{aligned}$$

Hence for $(g,h) \in U_{2-loc}$ we have using Lemma 3.4.7

$$\begin{aligned}
I_2(\omega)(g,h) &= \int_{\gamma_{g \rhd h}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g \rhd h}} ((g.\beta) - \beta)^{eq} \\
&= \int_{\gamma_{g \rhd h}} (g.\beta)^{eq} - \int_{\gamma_{g \rhd h}} \beta^{eq} \\
&= \int_{\gamma_{g \rhd h}} (c^*_{g^{-1}}(g.(\beta^{eq}))) - \int_{\gamma_{g \rhd h}} \beta^{eq} \\
&= \int_{c_{g^{-1}} \circ \gamma_{g \rhd h}} g.(\beta^{eq}) - \int_{\gamma_{g \rhd h}} \beta^{eq} \\
&= g. \int_{c_{g^{-1}} \circ \gamma_{g \rhd h}} \beta^{eq} - \int_{\gamma_{g \rhd h}} \beta^{eq} \\
&= g. \int_{\gamma_h} \beta^{eq} - \int_{\gamma_{g \rhd h}} \beta^{eq} \\
&= d_R(I^1(\beta))(g,h).
\end{aligned}$$

Hence $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ is sent to $BR^2_p(U, \mathfrak{a})_s$.

□

Proposition 3.4.9. *I^2 is a left inverse for δ^2.*

Proof : Let $x, y \in \mathfrak{g}$, and I_x (resp I_y) be an interval in $\mathbb{R}$ such that $\epsilon_x(s) = exp(sx)$ (resp $\epsilon_y(s) = exp(sy)$) be defined for all $s \in I_x$ (resp for all $s \in I_y$). The map $\epsilon_x \rhd \epsilon_y : I_x \times I_y \to G$ is continuous, thus there exists W an open subset of $I_x \times I_y$ such that $(\epsilon_x \rhd \epsilon_y)(W) \subseteq U$. Hence, there exists an interval $J \subseteq I_x \cap I_y$ such that $\epsilon_x(s) \rhd \epsilon_y(t) \in U$ for all $(s,t) \in J \times J$.

We have to show

$$\delta^2 \circ I^2 = id.$$

Let $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a)$. By definition, we have

$$\begin{aligned}
\delta^2(I^2(\omega))(x,y) &= \frac{\partial^2}{\partial s \partial t}\Big|_{|s,t=0} I^2(\omega)(\epsilon_x(s), \epsilon_y(s)) \\
&= \frac{\partial^2}{\partial s \partial t}\Big|_{|s,t=0} \int_{\gamma_{\epsilon_x(s) \rhd \epsilon_y(t)}} (I(\omega)(\epsilon_x(s)))^{eq} \\
&= \frac{\partial}{\partial s}\Big|_{|s=0} \Big(\frac{\partial}{\partial t}\Big|_{|t=0} \int_{\gamma_{\epsilon_y(t)}} c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s)))^{eq} \Big).
\end{aligned}$$

First, we compute

$$\frac{\partial}{\partial t}\Big|_{|t=0} \int_{\gamma_{\epsilon_y(t)}} c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s)))^{eq}$$

For the sake of clarity, we put $\alpha = c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s)))^{eq}$ and $\beta_t = \gamma_{\epsilon_y(t)}$. We have

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{|t=0} \int_\beta \alpha &= \frac{\partial}{\partial t}\Big|_{|t=0} \int_{[0,1]} \beta^* \alpha \\
&= \frac{\partial}{\partial t}\Big|_{|t=0} \int_{[0,1]} f_t(r)dr \\
&= \int_{[0,1]} \frac{\partial}{\partial t}\Big|_{|t=0} f_t(r)dr,
\end{aligned}$$

where $f_t(r) = \alpha(\beta_t(r))(\beta'_t(r))$.
We have

$$\frac{\partial}{\partial t}\Big|_{|t=0} f_t(r) = (\frac{\partial}{\partial t}\Big|_{|t=0} \alpha(\beta_t(r)))\beta'_0(r) + (\alpha(\beta_0(r)))(\frac{\partial}{\partial t}\Big|_{|t=0} \beta'_t(r)).$$

Moreover, we have

$$\alpha(\beta_0(r)) = \alpha(1),$$
$$\beta'_0(r) = 0,$$

and

$$\frac{\partial}{\partial t}\Big|_{|t=0} \beta'_t(r) = y.$$

So we have

$$\frac{\partial}{\partial t}\Big|_{|t=0} \int_\beta \alpha = \int_{[0,1]} \alpha(1)(x)dr = \alpha(1)(y)$$

and

$$\delta^2(I^2(\omega))(x,y) = \frac{\partial}{\partial s}\Big|_{|s=0} (c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s)))^{eq})(1)(y).$$

Furthermore we have

$$\begin{aligned}
c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s))^{eq})(1)(y) &= (I(\omega)(\epsilon_x(s)))^{eq}(c_{\epsilon_x(s)}(1))(Ad_{\epsilon_x(s)}(y)) \\
&= I(\omega)(\epsilon_x(s))(Ad_{\epsilon_x(s)}(y)) \\
&= (\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y)).
\end{aligned}$$

If we put $\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq} = \sigma(s)$ and $Ad_{\epsilon_x(s)}(y) = \lambda(s)$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_{|s=0} ((\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y))) &= \frac{\partial}{\partial s}\Big|_{|s=0} \sigma(s)(\lambda(s)) \\
&= \sigma'(0)(\lambda(0)) + \sigma(0)(\lambda'(0)).
\end{aligned}$$

We have

$$\sigma(0) = 0,$$
$$\lambda(0) = y,$$

and

$$\sigma'(0) = \tau^2(\omega)(x).$$

Thus

$$\frac{\partial}{\partial s}\Big|_{|s=0}((\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y))) = \tau^2(\omega)(x)(y).$$

Hence $\delta^2(I^2(\omega))(x,y) = \omega(x,y)$.

$\square$

Remark 3.4.10. Suppose that we have a Leibniz 2-cocycle ω which is also a Lie 2-cocycle. In this case, we can integrate ω into a local Lie rack cocycle, but also into a local Lie group cocycle. Then it is natural to ask if the two constructions are related to each other.

Proposition 3.4.11. *Let G be a Lie group, let $\mathfrak{g}$ be its Lie algebra, let $\mathfrak{a}$ be a representation of G, $\omega \in \Lambda^2(\mathfrak{g},\mathfrak{a})$ and γ_1, γ_2 smooth paths in G pointed in 1. Then*

$$\int_{\gamma_1}\Big(\int_{\gamma_2}(\tau^2(\omega))^{eq}\Big)^{eq} = \int_{\gamma_1\gamma_2}\omega^{eq}$$

where $\gamma_1\gamma_2 : [0,1]^2 \to G; (s,t) \mapsto \gamma_1(t)\gamma_2(s)$.

Proof : On the hand, we have

$$\begin{aligned}\int_{\gamma_1\gamma_2}\omega^{eq} &= \int_{[0,1]^2}(\gamma_1\gamma_2)^*\omega^{eq}\\ &= \int_{[0,1]^2}\omega^{eq}(\gamma_1\gamma_2)(\frac{\partial}{\partial s}\gamma_1(t)\gamma_2(s), \frac{\partial}{\partial t}\gamma_1(t)\gamma_2(s))dsdt,\end{aligned}$$

and this expression is equal to

$$\int_{[0,1]^2}\gamma_1(t)\gamma_2(s).\omega\big(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)), Ad_{\gamma_2(s)^{-1}}(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))\big). \tag{3.1}$$

On the other hand, we have

$$\begin{aligned}\int_{\gamma_1}\Big(\int_{\gamma_2}(\tau^2(\omega))^{eq}\Big)^{eq} &= \int_{[0,1]}\gamma_1^*(\int_{\gamma_2}(\tau^2(\omega)^{eq})^{eq})\\ &= \int_{[0,1]}(\int_{\gamma_2}(\tau^2(\omega))^{eq})^{eq}(\gamma_1(t))(\frac{\partial}{\partial t}\gamma_1(t))dt\\ &= \int_{[0,1]}\gamma_1(t).(\int_{\gamma_2}(\tau^2(\omega))^{eq})(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt\\ &= \int_{[0,1]}\gamma_1(t).(\int_{[0,1]}(\tau^2(\omega))^{eq}(\gamma_2(s))(\frac{\partial}{\partial s}\gamma_2(s)))(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt.\end{aligned}$$

This expression is equal to

$$\int_{[0,1]}\gamma_1(t).(\int_{[0,1]}\gamma_2(s).(\tau^2(\omega))(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)))ds)(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt,$$

which is equal to

$$\int_{[0,1]}\gamma_1(t).(\int_{[0,1]}\gamma_2(s).\omega(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)), Ad_{\gamma_2(s)^{-1}}(\cdot))ds)(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt.$$

Using the Fubini theorem, we show that this expression is equal to (3.1).

$\square$

If we apply this result to the case where $\gamma_1(s) = \gamma_{g\rhd h}(s) = exp(s\log(g\rhd h))$ and $\gamma_2(s) = \gamma_g(s) = exp(s\log(g))$ for $(g,h) \in U_{2-loc}$, then we obtain

Corollary 3.4.12. *If $\omega \in ZL^2(\mathfrak{g},\mathfrak{a}^a) \cap Z^2(\mathfrak{g},\mathfrak{a})$, then $I^2(\omega) = \Delta^2(\iota^2(\omega))$.*

We can remark that I^2 is more than a local Lie rack cocycle. Precisely, if ω is in $ZL^2(\mathfrak{g},\mathfrak{a}^a)$ then the local rack cocycle identity satisfied by $I^2(\omega)$, comes from another identity satisfied by $I^2(\omega)$. Indeed, I^2 is defined using I, and to verify that I^2 sends Leibniz cocycles into local rack cocycles, we have used Proposition 3.4.2. This proposition establishes that I^1 sends Lie cocycles into rack cocycle. But, we have remarked in Proposition 3.4.4 that the rack cocycle identity satisfied by $I^1(\omega)$, comes from the group cocycle identity. Hence, we can think that we forgot structure on $I^2(\omega)$. The following proposition identifies the identity satisfied by $I^2(\omega)$ which induced the local rack identity.

Proposition 3.4.13. *If $\omega \in ZL^2(\mathfrak{g},\mathfrak{a}^a)$, then $I^2(\omega)$ satisfies the identity*

$$g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g,h\rhd k) = 0, \quad \forall (g,h,k) \in U_{3-loc}.$$

Moreover, this identity induces the local rack cocycle identity.

Proof : Let $\omega \in ZL^2(\mathfrak{g},\mathfrak{a}^a)$ and $(g,h,k) \in U_{3-loc}$ we have:

$$\begin{aligned}
g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g,h\rhd k) &= g.\int_{\gamma_{h\rhd k}} (I(\omega)(h))^{eq} - \int_{\gamma_{(gh)\rhd k}} (I(\omega)(gh))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{h\rhd k}} g.((I(\omega)(h))^{eq}) - \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(gh))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{h\rhd k}} c_g^*((g.I(\omega)(h))^{eq}) - \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(gh))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq}
\end{aligned}$$

$$
\begin{aligned}
g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g,h\rhd k) &= \int_{c_g\circ\gamma_{h\rhd k}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(gh))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(gh))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h))^{eq} - (I(\omega)(gh))^{eq} + I(\omega)(g)^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h) - I(\omega)(gh) + I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} d(I(\omega))(g,h) \\
&= 0.
\end{aligned}
$$

Hence $I^2(\omega)$ satisfies the wanted identity.
Moreover, let $(g,h,k) \in U_{3-loc}$, we have

$$
\begin{aligned}
d_R(I^2(\omega))(g,h,k) &= g.(I^2(\omega)(h,k)) - I^2(\omega)(g\rhd h, g\rhd k) - (g\rhd h).I^2(\omega)(g,k) + I^2(\omega)(g,h\rhd k) \\
&= g.(I^2(\omega)(h,k)) - I^2(\omega)(g\rhd h, g\rhd k) - (g\rhd h).I^2(\omega)(g,k) + I^2(\omega)(g,h\rhd k) \\
&- I^2(\omega)(gh,k) + I^2(\omega)(gh,k) \\
&= \big(g.(I^2(\omega)(h,k)) - I^2(\omega)(gh,k) + I^2(\omega)(g,h\rhd k)\big) \\
&- \big((g\rhd h).I^2(\omega)(g,k) - I^2(\omega)(gh,k) + I^2(\omega)(g\rhd h, g\rhd k)\big) \\
&= 0 - 0 \\
&= 0.
\end{aligned}
$$

Hence $I^2(\omega)$ is a local Lie rack cocycle.

□

We will see in the next section that this identity make it possible to integrate a Leibniz algebra into a local augmented Lie rack.

3.5 From Leibniz algebras to local Lie racks

In this section, we present the main theorem of our thesis. In Proposition 3.1.1 we have seen that the tangent space at 1 of a (local) Lie rack is provided with a Leibniz algebra structure. Conversely, we now show that every Leibniz algebra can be integrated into an augmented local Lie rack. Our construction is explicit, and by this construction, a Lie algebra is integrated into a Lie group. Conversely, we show that an augmented local Lie rack whose tangent space at 1 is a Lie algebra is necessarily a (local) Lie group. That is, there is a structure of Lie group on this augmented local Lie rack, and the conjugation on the augmented local Lie rack is the conjugation in the group.

The idea of the proof is simple and uses the knowledge of the Lie's third theorem. Let $\mathfrak{g}$ be a Leibniz algebra. First, we decompose the vector space $\mathfrak{g}$ into a direct sum of Leibniz algebras $\mathfrak{g}_0$ and $\mathfrak{a}$ that we know how to integrate. As we know the theorem for Lie algebras, this is the case if $\mathfrak{g}$ is an abelian extension of a Lie algebra $\mathfrak{g}_0$ by a $\mathfrak{g}_0$-representation $\mathfrak{a}$. Hence, $\mathfrak{g}$ is isomorphic to $\mathfrak{a} \oplus_\omega \mathfrak{g}_0$. The Leibniz algebra $\mathfrak{a}$ is abelian, so becomes integrated into $\mathfrak{a}$, and $\mathfrak{g}_0$ is a Lie algebra, so becomes integrated into a simply connected Lie group G_0. Now, we have to understand how to patch $\mathfrak{a}$ and G_0. That is, we have to understand how the gluing data ω becomes integrated into a gluing data f between $\mathfrak{a}$ and G_0. It is the local Lie rack cocycle $I^2(\omega)$, constructed in the preceding section, which answers to this question. Hence, we showed that a Leibniz algebra $\mathfrak{g}$ becomes integrated into a local Lie rack of the form $\mathfrak{a} \times_f G_0$.

Let $\mathfrak{g}$ be a Leibniz algebra, there are several ways to see $\mathfrak{g}$ as an abelian extension of a Lie algebra $\mathfrak{g}_0$ by a $\mathfrak{g}_0$-representation $\mathfrak{a}$. Here, we take the abelian extension associated to the (left) center of $\mathfrak{g}$. By definition (see the end of section 1.3.3), the left center is

$$Z_L(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [x, y] = 0 \;\; \forall y \in \mathfrak{g}\}.$$

The left center $Z_L(\mathfrak{g})$ is an ideal in $\mathfrak{g}$ and we can consider the quotient of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$. It is a Leibniz algebra, and more precisely, a Lie algebra because $\mathfrak{g}_{ann}$ is included in $Z_L(\mathfrak{g})$. We denote this quotient by $\mathfrak{g}_0$. Hence, to a Leibniz algebra $\mathfrak{g}$ there is a canonical abelian extension given by

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0.$$

This extension gives a structure of $\mathfrak{g}_0$-representation to $Z_L(\mathfrak{g})$, and by definition of $Z_L(\mathfrak{g})$, this representation is anti-symmetric. The equivalence class of this extension is caracterised by a cohomology class in $HL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$. Hence by Theorem 1.3.13, there is $\omega \in ZL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$ such that the abelian extension $Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$ is equivalent to

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g}_0 \oplus_\omega Z_L(\mathfrak{g}) \overset{\pi}{\twoheadrightarrow} \mathfrak{g}_0.$$

Here $\mathfrak{g}_0$ is a Lie algebra, so becomes integrated into a simply connected Lie group G_0, and $Z_L(\mathfrak{g})$ is an abelian Lie algebra, so becomes integrated into itself. $Z_L(\mathfrak{g})$ is a $\mathfrak{g}_0$-representation (in the sense of Lie algebra) and G_0 is simply connected, thus by the Lie's second theorem, $Z_L(\mathfrak{g})$ is a smooth G_0-module (in the Lie group sense). Because of Proposition 2.3.23 and Example 2.2.5, $Z_L(\mathfrak{g})$ is provided with an anti-symmetric smooth G_0-module structure. The cocycle $\omega \in ZL^2(\mathfrak{g}, Z_L(\mathfrak{g}))$ becomes integrated into the local Lie rack cocycle $I^2(\omega) \in ZR^2_p(G_0, Z_L(\mathfrak{g}))_s$, and we can put on the cartesian product $G_0 \times Z_L(\mathfrak{g})$ a structure of local Lie rack by setting

$$(g, a) \rhd (h, b) = (g \rhd h, \phi_{g,h}(b) + \psi_{g,h}(a) + I^2(\omega)(g, h)),$$

where $\phi_{g,h}(b) = g.b$ and $\psi_{g,h}(a) = 0$. That is we have

$$(g, a) \rhd (h, b) = (g \rhd h, g.b + I^2(\omega)(g, h))).$$

It is clear by construction that this local Lie rack has its tangent space at 1 provided with a Leibniz algebra structure isomorphic to $\mathfrak{g}$. Finally, we showed the following theorem

Theorem 3.5.1. *Every Leibniz algebra $\mathfrak{g}$ can be integrated to a local Lie rack of the form*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a,$$

with conjugation

$$(g,a) \rhd (h,b) = (g \rhd h, g.b + I^2(\omega)(g,h)), \tag{3.2}$$

and neutral element $(1,0)$*, where* G_0 *is a Lie group,* $\mathfrak{a}$ *a* G_0*-module and* $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$*. Conversely, every local Lie rack of this form has its tangent space at* 1 *provides with a Leibniz algebra structure.*

Remark 3.5.2. Actually, we don't need the Lie's third theorem, we just need the Lie's first theorem and Lie's second theorem. Indeed, we use the Lie's third theorem to integrate $\mathfrak{g}_0$, but $\mathfrak{g}_0$ is isomorphic (by its adjoint representation) to a Lie subalgebra of $End(\mathfrak{g}_0)$. Hence, by the Lie's first theorem, there exists a Lie subgroup of $GL(\mathfrak{g}_0)$ which integrates $\mathfrak{g}_0$.

We ask more in our original problem. Indeed, we ask that a Lie algebra becomes integrated into a Lie group. That is, we have to show that when $\mathfrak{g}$ is a Lie algebra, then $G_0 \times Z_L(\mathfrak{g})$ is provided with a Lie group structure, and the conjugation on $G_0 \times_{I^2(\omega)} Z_L(\mathfrak{g})$ is induced by the rack product in $Conj(G_0 \times Z_L(\mathfrak{g}))$.

Let $\mathfrak{g}$ be a Lie algebra, the left center $Z_L(\mathfrak{g})$ is equal to the center $Z(\mathfrak{g})$. The abelian extension $Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$ provides $Z_L(\mathfrak{g})$ with an anti-symmetric structure but also a symmetric structure, so a trivial structure. This extension becomes a central extension and the cocycle $\omega \in ZL^2(\mathfrak{g}_0, Z(\mathfrak{g}))$ is also in $Z^2(\mathfrak{g}_0, Z(\mathfrak{g}))$. On the hand, with ω we can construct a local Lie rack cocycle $I^2(\omega)$, and on the other hand, we can construct a Lie group cocycle $\iota^2(\omega)$. In Corollary 3.4.12, we showed that $I^2(\omega) = \Delta^2(\iota^2(\omega))_{|U}$. Hence, the conjugation in $G_0 \times_{I^2(\omega)} Z(\mathfrak{g})$ can be written

$$(g,a) \rhd (h,b) = (g \rhd h, I^2(\omega)(g,h)) = (g \rhd h, \Delta^2(\iota^2(\omega))_{|U}(g,h)),$$

and we have seen in Remark 2.3.27 that this is the formula for the conjugation in the group $G_0 \times_{\iota^2(\omega)} Z(\mathfrak{g})$, where the product is defined by

$$(g,a)(h,b) = (gh, \iota^2(g,h)).$$

Conversely, suppose that a local Lie rack of the form $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ has its tangent space at 1, $\mathfrak{g}_0 \oplus_\omega \mathfrak{a}^a$, provided with a Lie algebra structure. Necessarily, $\mathfrak{a}$ is a trivial $\mathfrak{g}_0$-representation and $\omega \in Z^2(\mathfrak{g}_0, \mathfrak{a})$. Hence, as before, $I^2(\omega) = \Delta^2(\iota^2(\omega))_{|U}$ and the conjugation defined by the formula (3.2) is induced by the conjugation coming from the group structure on $G_0 \times_{\iota^2(\omega)} \mathfrak{a}$. Finally, we have the following theorem

Theorem 3.5.3. *Every Leibniz algebra $\mathfrak{g}$ becomes integrated into a local Lie rack of the form*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a,$$

with conjugation

$$(g,a) \rhd (h,b) = (g \rhd h, g.b + I^2(\omega)(g,h)), \tag{3.3}$$

and neutral element $(1,0)$*, where* G_0 *is a Lie group,* $\mathfrak{a}$ *a representation of* G_0 *and* $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$*. Conversely, every local Lie rack of this form has its tangent space at* 1 *provided with a Leibniz algebra structure.*

Moreover, in the special case where $\mathfrak{g}$ is a Lie algebra, the above construction provides $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ with a rack product coming from the conjugation in a Lie group. Conversely, if the tangent space at 1 of $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ is a Lie algebra, then $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ can be provided with a Lie group structure, and the conjugation induced by the Lie group structure is the one defined by (3.3).

3.6 From Leibniz algebras to local augmented Lie racks

Let $\mathfrak{g}_0$ be a Lie algebra, $\mathfrak{a}$ a $\mathfrak{g}$-representation and $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$. In Proposition 3.4.5, we showed that $I^2(\omega)$ is a local Lie rack cocycle. We showed also that it satisfies the identity

$$g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g, h \rhd k) = 0 \tag{3.4}$$

for all $(g,h,k) \in U_{3-loc}$.
The natural question is : What is the algebraic structure on $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ encoded by this identity?

Definition 3.6.1. *Let G be a group. A **local** G**-set** is a set X provides with a map ρ defined on a subset Ω of $G \times X$ with values in X such that the followings axioms are satisfied*

1. *If $(h,x), (gh,x), (g, \rho(h,x)) \in \Omega$, then $\rho(g, \rho(h,x)) = \rho(gh, x)$.*
2. *For all $x \in X$, we have $(1,x) \in \Omega$ and $\rho(1,x) = x$.*

*A **local topological** (resp.(**smooth**)) G**-set** is a topological set (resp. smooth manifold) X with a structure of a local G-set such that*

1. *Ω is an open subset of X.*
2. *$\rho : \Omega \to X$ is continuous (resp. smooth).*

*A **fixed point** is an element $x_0 \in X$ such that for all $g \in G, (g, x_0) \in \Omega$ and $\rho(g, x_0) = x_0$.*

In the following proposition, we show that the identity (3.4) provides $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ with a structure of a local G_0-set.

Proposition 3.6.2. *$G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ is a local smooth G_0-set, and $(1,0)$ is a fixed point.*

Proof : We define an open subset Ω and a smooth map ρ by

1. $\Omega = \{(g,(h,b)) \in G_0 \times (G_0 \times_{I^2(\omega)} \mathfrak{a}^a) | (g,h) \in U_{2-loc}\}$.
2. $\rho(g,(h,b)) = (g \rhd h, g.b + I^2(\omega)(g,h))$.

Let $(h,(k,z)), (gh,(k,z)), (g, \rho(h,(k,z))) \in \Omega$. This is equivalent to the condition $(h,k), (gh,k), (g, h \rhd k) \in U_{2-loc}$, that is $(g,h,k) \in U_{3-loc}$. We have

$$\begin{aligned}\rho(g, \rho(h,(k,z))) &= \rho(g, (h \rhd k, h.z + I^2(\omega)(h,k))) \\ &= (g \rhd (h \rhd k), g.(h.z) + g.I^2(\omega)(h,k) + I^2(\omega)(g, h \rhd k)).\end{aligned}$$

Using the identities (2) and $(gh) \rhd k = g \rhd (h \rhd k)$, we have

$$\begin{aligned}\rho(g, \rho(h,(k,z))) &= ((gh) \rhd k, (gh).z + I^2(\omega)(gh,k)) \\ &= \rho(gh, \rho(k,z)).\end{aligned}$$

Thus $\rho(g, \rho(h,(k,z))) = \rho(gh, \rho(k,z))$.
Moreover, we have $\rho(1,(k,z)) = (1 \rhd k, 1.z + I^2(\omega)(1,k)) = (k,z)$ and $\rho(g,(1,0)) = (g \rhd 1, g.0 + I^2(\omega)(g,1)) = (1,0)$. Hence $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ is a local smooth G_0-set and $(1,0)$ is a fixed point for this local action.

$\square$

We remark that we can reconstruct the rack product in $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ from the formula of the G_0-action. Indeed, we have

$$(g, a) \rhd (h, b) = g.(h, b) = p(g, a).(h, b),$$

where p is the projection on the first factor $G_0 \times_{I^2(\omega)} \mathfrak{a}^a \overset{p}{\twoheadrightarrow} G_0$. Furthermore, $p(1, 0) = 1$ and p is equivariant. Indeed, let $(g, (h, y)) \in \Omega$, we have $p(\rho(g.(h, y))) = p(g \rhd h, g.y + I^2(\omega)(g, h)) = g \rhd h = g.p(h, y)$. Hence we showed the following proposition

Proposition 3.6.3. *$G_0 \times_{I^2(\omega)} \mathfrak{a}^a \overset{p}{\twoheadrightarrow} G_0$ is a local augmented Lie rack.*

Hence we can rewrite our main theorem

Theorem 3.6.4. *Every Leibniz algebra $\mathfrak{g}$ becomes integrated into a local augmented Lie rack of the form*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a \overset{p}{\twoheadrightarrow} G_0,$$

with local action

$$g.(h, b) = (g \rhd h, g.b + I^2(\omega)(g, h)),$$

and neutral element $(1, 0)$, where G_0 is a Lie group, $\mathfrak{a}$ a representation of G_0 and $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$. Conversely, every local augmented Lie rack of this form has its tangent space at 1 provided with a Leibniz algebra structure.

Moreover, in the special case where $\mathfrak{g}$ is a Lie algebra, the above construction provides $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ with a rack product coming from the conjugation in a Lie group. Conversely, if the tangent space at 1 of $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ is a Lie algebra, then $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ can be provided with a Lie group structure, and the conjugation induced by the Lie group structure is the one defined by (3.3).

3.7 Examples of non split Leibniz algebra integrations

3.7.1 In dimension 4

Example 3.7.1. Let $\mathfrak{g} = \mathbb{R}^4$. We define a bilinear map on $\mathfrak{g}$ by

$$\begin{aligned}
[e_1, e_1] &= e_4 \\
[e_1, e_2] &= e_4 \\
[e_2, e_1] &= -e_4 \\
[e_2, e_2] &= e_4 \\
[e_3, e_3] &= e_4
\end{aligned}$$

We have

$$[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)] = (0, 0, 0, x_1 y_1 + x_1 y_2 - x_2 y_1 + x_2 y_2 + x_3 y_3)$$

Proposition 3.7.2. $(\mathfrak{g}, [-, -])$ *is a Leibniz algebra.*

Proof : We have

$$[(x_1,x_2,x_3,x_4),[(y_1,y_2,y_3,y_4),(z_1,z_2,z_3,z_4)]] = (0,0,0,0),$$

and

$$[(y_1,y_2,y_3,y_4),[(x_1,x_2,x_3,x_4),(z_1,z_2,z_3,z_4)]] = (0,0,0,0),$$

and

$$[[(x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)],(z_1,z_2,z_3,z_4)] = (0,0,0,0).$$

Hence the bracket $[-,-]$ satisfies the Leibniz identity.

□

To follow the method explained above, we have to determine the left center $Z_L(\mathfrak{g})$, the quotient $\mathfrak{g}_0$ of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$ denoted $\mathfrak{g}_0$, the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$ and the Leibniz 2-cocycle describing the abelian extension $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Let $x \in Z_L(\mathfrak{g})$, for $y = (1,0,0,0), y = (0,1,0,0)$ or $y = (0,0,1,0)$ in $\mathfrak{g}$, we have $[x,y] = 0$. This implies that $x_1 = x_2 = x_3 = 0$. Conversely, every element in $\mathfrak{g}$ with the first three coordinates equal to 0 is in $Z_L(\mathfrak{g})$. Hence $Z_L(\mathfrak{g}) =< e_4 >$ and $\mathfrak{g}_0 \simeq< e_1, e_2, e_3 >$. The bracket on $\mathfrak{g}_0$ is equal to zero, hence $\mathfrak{g}_0$ is an abelian Lie algebra. The action of $\mathfrak{g}_0$ on $ZL(\mathfrak{g})$ is given by

$$\rho_x(y) = [(x_1,x_2,x_3,0),(0,0,0,y_4)] = (0,0,0,0),$$

and the Leibniz 2-cocycle is given by

$$\omega(x,y) = [(x_1,x_2,x_3,0),(y_1,y_2,y_3,0)] = (0,0,0,x_1y_1 + x_1y_2 - x_2y_1 + x_2y_2 + x_3y_3).$$

Now, we have to determine the Lie group G_0 associated to $\mathfrak{g}_0$, the action of G_0 on $Z_L(\mathfrak{g})$ integrating $\rho : \mathfrak{g}_0 \to End(Z_L(\mathfrak{g}))$ (the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$), and the Lie rack cocycle integrating ω.

The Lie algebra $\mathfrak{g}_0$ is abelian, thus a Lie group integrating $\mathfrak{g}_0$ is $G_0 = \mathfrak{g}_0$. Moreover, ρ is null hence the Lie group action of G_0 on $Z_L(\mathfrak{g})$ which integrates ρ is the trivial action. The facts that ρ is zero and $\mathfrak{g}_0$ abelian, imply that $d^1_L : CL^1(\mathfrak{g}_0, Z_L(\mathfrak{g})) \to CL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$ is zero. Hence, because $\mathfrak{g}_{ann} = Z_L(\mathfrak{g})$, $\mathfrak{g}$ is non split. That is, $\mathfrak{g}$ is not isomorphic to the direct sum of a Lie algebra $\mathfrak{h}$ and a Lie representation V over $\mathfrak{h}$ provided with the bracket $[(x,v),(y,w)] = ([x,y], x.w)$ (see the introduction). What remains to be done is the integration of the cocycle ω. A formula for f, a Lie rack cocycle integrating ω, is

$$f(a,b) = \int_{\gamma_b} (\int_{\gamma_a} \tau^2(\omega)^{eq})^{eq},$$

where $\gamma_a(s) = sa$ and $\gamma_b(t) = tb$. Let $a \in G_0$ and $x, y \in \mathfrak{g}_0$. We have

$$\begin{aligned}
\int_{\gamma_a} \tau^2(\omega)^{eq} &= \int_{[0,1]} \tau^2(\omega)^{eq}(\gamma_a(s))(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_a(s))ds \\
&= \int_{[0,1]} \Phi_{\gamma_a(s)}(\tau^2(\omega)(a))ds \\
&= \int_{[0,1]} \tau^2(\omega)(a)ds \\
&= \tau^2(\omega)(a).
\end{aligned}$$

Thus

$$
\begin{aligned}
f(a,b) &= \int_{\gamma_b} (\tau^2(\omega)(a))^{eq} \\
&= \int_{[0,1]} \gamma_b^*(\tau^2(\omega)(a))^{eq} \\
&= \int_{[0,1]} \tau^2(\omega)(a))^{eq}(\gamma_b(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_b(t))dt \\
&= \int_{[0,1]} \tau^2(\omega)(a))(b)dt \\
&= \tau^2(\omega)(a)(b) \\
&= \omega(a,b).
\end{aligned}
$$

Hence, the conjugation in $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^4$ is given by

$$
(a_1,a_2,a_3,a_4) \triangleright (b_1,b_2,b_3,b_4) = (b_1,b_2,b_3,a_1b_1 + a_2b_2 + a_3b_3 + a_1b2 - a_2b_1 + b_4).
$$

We have

$$
\begin{aligned}
\frac{\partial^2}{\partial s \partial t}_{|s,t=0}(sa_1,sa_2,sa_3,sa_4) \triangleright (tb_1,tb_2,tb_3,tb_4) &= \frac{\partial^2}{\partial s \partial t}_{|s,t=0}(tb_1,tb_2,tb_3,st(a_1b_1 + a_2b_2 + a_3b_3 + a_1b_2 - a_2b_1),tb_4) \\
&= (0,0,0,a_1b_1 + a_2b_2 + a_3b_3 + a_1b_2 - a_2b_1) \\
&= [(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4)].
\end{aligned}
$$

Thus $(\mathbb{R}^4, \triangleright)$ integrates $(\mathbb{R}^4, [-,-])$.

Example 3.7.3. Let $\mathfrak{g} = \mathbb{R}^4$. We define a bilinear map on $\mathfrak{g}$ by

$$
\begin{aligned}
[e_1,e_1] &= e_2 \\
[e_1,e_2] &= e_3 \\
[e_1,e_3] &= e_4
\end{aligned}
$$

We have

$$
[(x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)] = (0,x_1y_1,x_1y_2,x_1y_3)
$$

Proposition 3.7.4. *$(\mathfrak{g},[-,-])$ is a Leibniz algebra.*

Proof : We have

$$
\begin{aligned}
[(x_1,x_2,x_3,x_4),[(y_1,y_2,y_3,y_4),(z_1,z_2,z_3,z_4)]] &= [(x_1,x_2,x_3,x_4),(0,y_1z_1,y_1z_2,y_1z_3)] \\
&= (0,0,x_1y_1z_2,x_1y_1z_3),
\end{aligned}
$$

and

$$
[(y_1,y_2,y_3,y_4),[(x_1,x_2,x_3,x_4),(z_1,z_2,z_3,z_4)]] = (0,0,x_1y_1z_2,x_1y_1z_3),
$$

and

$$
[[(x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)],(z_1,z_2,z_3,z_4)] = 0.
$$

Hence the bracket $[-,-]$ satisfies the Leibniz identity.

$\square$

To follow the method explained above, we have to determine the left center $Z_L(\mathfrak{g})$, the quotient $\mathfrak{g}_0$ of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$ denoted $\mathfrak{g}_0$, the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$ and the Leibniz 2-cocycle describing the abelian extension $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Let $x \in Z_L(\mathfrak{g})$, for $y = (1,0,0,0), y = (0,1,0,0)$ or $y = (0,0,1,0)$ in $\mathfrak{g}$, we have $[x,y] = 0$. This implies that $x_1 = 0$. Conversely, every element in $\mathfrak{g}$ with the first coordinate equals to 0 is in $Z_L(\mathfrak{g})$. Hence $Z_L(\mathfrak{g}) =< e_2, e_3, e_4 >$ and $\mathfrak{g}_0 \simeq< e_1 >$. The bracket on $\mathfrak{g}_0$ is equal to zero, hence $\mathfrak{g}_0$ is an abelian Lie algebra. The action of $\mathfrak{g}_0$ on $ZL(\mathfrak{g})$ is given by

$$\rho_x(y) = [(x_1,0,0,0),(0,y_2,y_3,y_4)] = (0,0,x_1y_2,x_1y_3),$$

and the Leibniz 2-cocycle is given by

$$\omega(x,y) = [(x_1,0,0,0),(y_1,0,0,0)] = (0,x_1y_1,0,0).$$

Moreover, we have $[x,x] = (0,x_1^2,x_1x_2,x_1x_3)$, hence $\mathfrak{g}_{ann} = Z_L(\mathfrak{g})$. This Leibniz algebra is not split because for $\alpha \in Hom(\mathfrak{g}_0,\mathbb{R})$ and $x,y \in \mathfrak{g}_0$, we have $d_L\alpha(x,y) = \rho_x(\alpha(y)) = (0,0,x_1\alpha(y)_2,x_1\alpha(y)_3)$.

Now, we have to determine the Lie group G_0 associated to $\mathfrak{g}_0$, the action of G_0 on $Z_L(\mathfrak{g})$ integrating $\rho : \mathfrak{g}_0 \to End(Z_L(\mathfrak{g}))$ (the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$), and the Lie rack cocycle integrating ω.

The Lie algebra $\mathfrak{g}_0$ is abelian, thus a Lie group integrating $\mathfrak{g}_0$ is $G_0 = \mathfrak{g}_0$. Moreover, a simple calculation shows that the Lie group action of G_0 on $Z_L(gg)$ defined by

$$\phi_x(y) = y + \rho_x(y),$$

integrates ρ. What remains to be done is the integration of the cocycle ω. A formula for f, a Lie rack cocycle integrating ω, is

$$f(a,b) = \int_{\gamma_b}(\int_{\gamma_a}\tau^2(\omega)^{eq})^{eq},$$

where $\gamma_a(s) = sa$ and $\gamma_b(t) = tb$. Let $a \in G_0$ and $x, y \in \mathfrak{g}_0$. We have

$$\begin{aligned}
\int_{\gamma_a} \tau^2(\omega)^{eq} &= \int_{[0,1]} \tau^2(\omega)^{eq}(\gamma_a(s))(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_a(s))ds \\
&= \int_{[0,1]} \Phi_{\gamma_a(s)}(\tau^2(\omega)(a))ds \\
&= \int_{[0,1]} \phi_{\gamma_a(s)} \circ \tau^2(\omega)(a)ds \\
&= \int_{[0,1]} \begin{pmatrix} 0 \\ a \\ sa^2 \\ 0 \end{pmatrix} ds \\
&= \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ a^2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
f(a,b) &= \int_{\gamma_b} (\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix})^{eq} \\
&= \int_{[0,1]} \gamma_b^*(\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix})^{eq}.
\end{aligned}$$

$$\begin{aligned}
f(a,b) &= \int_{[0,1]} \begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix}^{eq} (\gamma_b(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_b(t))dt \\
&= \int_{[0,1]} \phi_{\gamma_b(t)}(\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix}(b))dt \\
&= \int_{[0,1]} abe_2 + (tab^2 + \frac{1}{2}a^2b)e_3 + \frac{1}{2}ta^2b^2e_4 dt \\
&= abe_2 + \frac{1}{2}(ab^2 + a^2b)e_3 + \frac{1}{4}a^2b^2e_4.
\end{aligned}$$

Hence, the conjugation in $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^4$ is given by

$$(a_1, a_2, a_3, a_4) \triangleright (b_1, b_2, b_3, b_4) = (b_1, a_1b_1, a_1b_2 + \frac{1}{2}(a_1b_1^2 + a_1^2b_1), a_1b_3 + \frac{1}{4}a_1^2b_1^2).$$

We have

$$\frac{\partial^2}{\partial s \partial t}_{|s,t=0}(sa_1, sa_2, sa_3, sa_4) \rhd (tb_1, tb_2, tb_3, tb_4) = \frac{\partial^2}{\partial s \partial t}_{|s,t=0}(tb_1, sta_1b_1, sta_1b_2 + \frac{1}{2}(st^2a_1b_1^2 + s^2ta_1^2b_1),$$
$$sta_1b_3 + s^2t^2\frac{1}{4}a_1^2b_1^2)$$
$$= (0, a_1b_1, a_1b_2, a_1b_3)$$
$$= [(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)].$$

Thus $(\mathbb{R}^4, \rhd)$ integrates $(\mathbb{R}^4, [-,-])$.

3.7.2 In dimension 5

Example 3.7.5. Let $\mathfrak{g} = \mathbb{R}^5$. We define a bilinear map on $\mathfrak{g}$ by

$$[e_1, e_1] = [e_1, e_2] = e_3$$
$$[e_2, e_1] = [e_2, e_2] = [e_1, e_3] = e_4$$
$$[e_1, e_4] = [e_2, e_3] = e_5$$

We have

$$[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)] = (0, 0, x_1(y_1 + y_2), x_2(y_1 + y_2) + x_1y_3, x_1y_4 + x_2y_3).$$

Proposition 3.7.6. $(\mathfrak{g}, [-,-])$ *is a Leibniz algebra.*

Proof : We have

$$[(x_1, x_2, x_3, x_4), [(y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, x_1y_1(z_1+z_2), x_1(y_2(z_1+z_2)+y_1z_3)+x_2y_1(z_1+z_2)),$$

and

$$[(y_1, y_2, y_3, y_4), [(x_1, x_2, x_3, x_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, x_1y_1(z_1+z_2), x_1(y_2(z_1+z_2)+y_1z_3)+x_2y_1(z_1+z_2)),$$

and

$$[[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)], (z_1, z_2, z_3, z_4)] = (0, 0, 0, 0).$$

Hence the bracket $[-,-]$ satisfies the Leibniz identity.

□

To follow the method explained above, we have to determine the left center $Z_L(\mathfrak{g})$, the quotient of $\mathfrak{g}$ by $Z_L(\mathfrak{g})$ denoted $\mathfrak{g}_0$, the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$ and the Leibniz 2-cocycle describing the abelian extension $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Let $x \in Z_L(\mathfrak{g})$, for $y = (0, 0, 1, 0, 0)$ in $\mathfrak{g}$, we have $[x, y] = 0$. This implies that $x_1 = x_2 = 0$. Conversely, every element in $\mathfrak{g}$ with the first two coordinates equal to 0 is in $Z_L(\mathfrak{g})$. Hence $Z_L(\mathfrak{g}) =< e_3, e_4, e_5 >$ and $\mathfrak{g}_0 \simeq< e_1, e_2 >$. The bracket on $\mathfrak{g}_0$ is equal to zero, hence $\mathfrak{g}_0$ is an abelian Lie algebra. The action of $\mathfrak{g}_0$ on $ZL(\mathfrak{g})$ is given by

$$\rho_x(y) = [(x_1, x_2, 0, 0, 0), (0, 0, y_3, y_4, y_5)] = (0, 0, 0, x_1y_3, x_1y_4 + x_2y_3),$$

and the Leibniz 2-cocycle is given by

$$\omega(x, y) = [(x_1, x_2, 0, 0, 0), (y_1, y_2, 0, 0, 0)] = (0, 0, x_1(y_1 + y_2), x_2(y_1 + y_2), 0).$$

Moreover, we have $[x,x]=(0,0,x_1(x_1+x_2),x_2(x_1+x_2)+x_1x_3,x_1x_4+x_2x_3)$, hence taking $x=(1,0,0,0,0),(0,1,0,0,0)$ and $(0,1,1,0,0)$, we see easily that $\mathfrak{g}_{ann}=Z_L(\mathfrak{g})$. This Leibniz algebra is not split because for $\alpha\in Hom(\mathfrak{g},Z_L(\mathfrak{g}))$ and $x,y\in\mathfrak{g}_0$, we have $d_L\alpha(x,y)=\rho_x(\alpha(y))=(0,0,0x_1\alpha(y)_3,x_1\alpha(y)_4+x_2\alpha(y)_3)$.

Now, we have to determine the Lie group G_0 associated to $\mathfrak{g}_0$, the action of G_0 on $Z_L(\mathfrak{g})$ integrating $\rho:\mathfrak{g}_0\to End(Z_L(\mathfrak{g}))$ (the action of $\mathfrak{g}_0$ on $Z_L(\mathfrak{g})$), and the Lie rack cocycle integrating ω.

The Lie algebra $\mathfrak{g}_0$ is abelian, thus a Lie group integrating $\mathfrak{g}_0$ is $G_0=\mathfrak{g}_0$. To integrate the action ρ, we use the exponential $exp: End(Z_L(\mathfrak{g}))\to Aut(Z_L(\mathfrak{g}))$. Indeed, for all $x\in\mathfrak{g}_0$, we have

$$\rho_x=\begin{pmatrix}0&0&0\\x_1&0&0\\x_2&x_1&0\end{pmatrix}.$$

Hence, we define a Lie group morphism $\phi:G_0\to Aut(Z_L(\mathfrak{g}))$ by setting

$$\phi_x=exp(\rho_x)=\begin{pmatrix}1&0&0\\x_1&1&0\\x_2+\frac{1}{2}x_1^2&x_1&0\end{pmatrix}.$$

It is easy to see that $d_1\phi=\rho$. What remains to be done is the integration of the cocycle ω. A formula for f, a Lie rack cocycle integrating ω, is

$$f(a,b)=\int_{\gamma_b}(\int_{\gamma_a}\tau^2(\omega)^{eq})^{eq},$$

where $\gamma_a(s)=sa$ and $\gamma_b(t)=tb$. Let $a\in G_0$ and $x,y\in\mathfrak{g}_0$. We have

$$\begin{aligned}\int_{\gamma_a}\tau^2(\omega)^{eq}&=\int_{[0,1]}\tau^2(\omega)^{eq}(\gamma_a(s))(\frac{\partial}{\partial s}\Big|_{s=0}\gamma_a(s))ds\\&=\int_{[0,1]}\Phi_{\gamma_a(s)}(\tau^2(\omega)(a))ds\\&=\int_{[0,1]}\phi_{\gamma_a(s)}\circ\tau^2(\omega)(a)ds\\&=\int_{[0,1]}\begin{pmatrix}1&0&0\\sa_1&1&0\\sa_2+\frac{1}{2}(sa_1)^2&sa_1&0\end{pmatrix}\begin{pmatrix}a_1&a_1\\a_2&a_2\\0&0\end{pmatrix}ds\\&=\int_{[0,1]}\begin{pmatrix}a_1&a_1\\sa_1^2+a_2&sa_1^2+a_2\\2sa_1a_2+\frac{1}{2}s^2a_1^3&2sa_1a_2+\frac{1}{2}s^2a_1^3\end{pmatrix}ds.\end{aligned}$$

Thus

$$\int_{\gamma_a}\tau^2(\omega)^{eq}=\begin{pmatrix}a_1&a_1\\\frac{1}{2}a_1^2+a_2&\frac{1}{2}a_1^2+a_2\\a_1a_2+\frac{1}{6}a_1^3&a_1a_2+\frac{1}{6}a_1^3\end{pmatrix}.$$

Hence

$$
\begin{aligned}
f(a,b) &= \int_{\gamma_b} (\int_{\gamma_a} \tau^2(\omega)^{eq})^{eq} \\
&= \int_{\gamma_b} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq} \\
&= \int_{[0,1]} \gamma_b^* \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq} \\
&= \int_{[0,1]} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq} (\gamma_b(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_b(t))dt \\
&= \int_{[0,1]} \phi_{\gamma_b(t)}(\begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}(b))dt.
\end{aligned}
$$

We have

$$
\begin{aligned}
\phi_{\gamma_b(t)}(\begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}(b)) &= \begin{pmatrix} 1 & 0 & 0 \\ tb_1 & 1 & 0 \\ tb_2 + \frac{1}{2}(tb_1)^2 & tb_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1(b_1+b_2) \\ (tb_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1+b_2) \\ (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{2}tb_1a_1^2 + tb_2a_1 + tb_1a_2 + \frac{1}{2}(tb_1)^2a_1)(b_1+b_2) \end{pmatrix}.
\end{aligned}
$$

Thus

$$
f(a,b) = \begin{pmatrix} a_1(b_1+b_2) \\ (\frac{1}{2}b_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1+b_2) \\ (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{4}b_1a_1^2 + \frac{1}{2}b_2a_1 + \frac{1}{2}b_1a_2 + \frac{1}{6}(b_1)^2a_1)(b_1+b_2) \end{pmatrix}.
$$

and the conjugation in $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^5$ is given by

$$
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} \rhd \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 + a_1(b_1+b_2) \\ a_1b_3 + b_4 + (\frac{1}{2}b_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1+b_2) \\ (a_2 + \frac{1}{2}a_1^2)b_3 + a_1b_4 + b_5 + (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{4}b_1a_1^2 + \frac{1}{2}b_2a_1 + \frac{1}{2}b_1a_2 + \frac{1}{6}(b_1)^2a_1)(b_1+b_2) \end{pmatrix}.
$$

With a simple computation we verify that

$$
\frac{\partial^2}{\partial s \partial t}\Big|_{s,t=0} (sa_1, sa_2, sa_3, sa_4, sa_5) \rhd (tb_1, tb_2, tb_3, tb_4, tb_5) = [(a_1,a_2,a_3,a_4,a_5),(b_1,b_2,b_3,b_4, b_5)].
$$

Thus $(\mathbb{R}^5, \rhd)$ integrates $(\mathbb{R}^5, [-,-])$.

Appendix A

Trunks

Our main reference on this subject is [FRS95].

A.1 Introduction

Let G be a group, one way to associate to G a category is to take

$$\begin{aligned} &\textit{Object} : g \in G \\ &\textit{Morphism} : g \xrightarrow{h} gh \\ &\textit{Composition} : (g \xrightarrow{h} gh) \circ (gh \xrightarrow{k} ghk) = g \xrightarrow{hk} ghk \\ &\textit{Identity} : g \xrightarrow{1} g \end{aligned}$$

We denote this category by $\mathcal{C}_G(G)$.

Remark A.1.1. We can generalize $\mathcal{C}_G(G)$ substituting G by any right G-module A. The objects become the elements of A, the morphisms are $a \xrightarrow{g} a.g$, the composition is $a \xrightarrow{g} a.g \circ a.g \xrightarrow{h} (a.g).h = a \xrightarrow{gh} a.gh$, and the identity is $a \xrightarrow{1} a$. We recover the case of $\mathcal{C}_G(G)$ when the structure of right G-module on G is defined by the right multiplication.

It is natural to wonder what is the correct notion when we replace the group structure by a rack structure. The correct notion is the notion of a trunk.

A.2 Definitions and examples

Definition A.2.1. *An* ***oriented square*** *in a directed graph is a sequence of edges a, b, c, d which we represent:*

$$\begin{tikzcd} C \arrow[r, "d"] & D \\ A \arrow[u, "c"] \arrow[r, "a"'] & B \arrow[u, "b"'] \end{tikzcd}$$

Definition A.2.2. *A* ***trunk*** *is a directed graph Γ, together with a collection of oriented squares in Γ called* ***preferred squares****.*

Definition A.2.3. *A **pointed trunk** is a trunk together with a chosen edge $e_A : A \to A$ for each vertex A, and with the following preferred squares for each edge $a : A \to B$.*

$$\begin{tikzcd} A \arrow[r, "a"] & B \\ A \arrow[u, "e_A"] \arrow[r, "a"'] & B \arrow[u, "e_B"'] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "e_B"] & B \\ A \arrow[u, "a"] \arrow[r, "e_A"'] & A \arrow[u, "a"'] \end{tikzcd}$$

*The edges e_A are called the **identities**.*

Definition A.2.4. *A **corner trunk** is a trunk which satisfies the following two corner axioms:*

($C1$) *Given edges $a : A \to B$ and $b : B \to D$, there are unique edges $a \rhd b : A \to C$ and $a \lhd b : C \to D$ such that the following square is preferred*

$$\begin{tikzcd} C \arrow[r, "a \lhd b"] & D \\ A \arrow[u, "a \rhd b"] \arrow[r, "a"'] & B \arrow[u, "b"'] \end{tikzcd}$$

($C2$) *In the following diagram, if the squares $(ABCD), (BDYT)$ and $(CDZT)$ are preferred, then the diagram can be completed as shown so that the squares $(ABXY), (ACXZ)$ and $(XYZT)$ are preferred.*

$$\begin{tikzcd} & Z \arrow[rr] & & T \\ X \arrow[ur, dotted] \arrow[rr, dotted] & & Y \arrow[ur] & \\ & C \arrow[uu] \arrow[rr] & & D \arrow[uu, "c"'] \\ A \arrow[uu, dotted] \arrow[ur] \arrow[rr, "a"'] & & B \arrow[uu] \arrow[ur, "b"'] & \end{tikzcd}$$

Given ($C1$), an equivalent statement is that the edges a, b and c determine the entire diagram of preferred squares.

Proposition A.2.5. *In a corner trunk, the binary operations $\rhd$ and $\lhd$ satisfy:*

1. $a \rhd (b \rhd c) = (a \rhd b) \rhd ((a \lhd b) \rhd c)$
2. $(a \lhd b) \lhd c) = (a \lhd (b \rhd c)) \lhd (b \lhd c)$
3. $(a \lhd (b \rhd c)) \rhd (b \lhd c) = (a \rhd b) \lhd ((a \lhd b) \rhd c)$

Definition A.2.6. *Let $\mathcal{S}$ and $\mathcal{T}$ two trunks. A **trunk map** between $\mathcal{T}$ and $\mathcal{S}$ maps vertices on vertices, edges on edges and preferred squares on preferred squares. If $\mathcal{S}$ and $\mathcal{T}$ are pointed trunks, then a **pointed trunk map** between $\mathcal{T}$ and $\mathcal{S}$ is a trunk map from $\mathcal{T}$ to $\mathcal{S}$ which takes identities to identities.*

Example A.2.7 (Category)**.** Let $\mathcal{C}$ be a category. We define a directed graph $\Gamma(\mathcal{C})$ using $\mathcal{C}$ by taking the objects as vertices, and the morphisms as edges. Hence, we define a trunk $Trunk(\mathcal{C})$ by taking the commutative squares in $\mathcal{C}$ as preferred squares. Moreover, this trunk is a pointed trunk using the identity morphisms in $\mathcal{C}$.

This construction gives a functor between the category of categories and the category of (pointed) trunks. Conversely, we can define a functor from the category of (pointed) trunks to the category of categories. We take as objects the vertices, and the morphisms are generated by the edges of the trunk with the relations which follows from insisting that preferred squares commute (and identifying the identities in the trunk with identites in the category for the pointed version). Then, this functor, denoted Cat, is left adjoint to the functor $Trunk$.

Example A.2.8 (The rack trunk)**.** This is the key example. Let X be a rack, we define a corner trunk $\mathcal{T}(X)$ by setting

Vertices: $*$

Edges: for all $x \in X$, $\ * \xrightarrow{x} *$

Preferred squares: for all $x, y \in X$,

$$\begin{tikzcd} * \arrow[r, "x"] & * \\ * \arrow[u, "x \triangleright y"] \arrow[r, "x"'] & * \arrow[u, "y"'] \end{tikzcd}$$

The corner trunk axioms are satisfied because of the axioms defining a rack. Indeed, by definition of this trunk, the axiom $(C1)$ is clear, and, because of the rack identity, the axiom $(C2)$ is satisfied. Moreover, if we suppose that X is a pointed rack, then $\mathcal{T}(X)$ is a pointed corner trunk with identity 1.

Example A.2.9 (The augmented rack trunk)**.** Let $X \xrightarrow{p} G$ be an augmented rack, we define a corner trunk $\mathcal{T}_G(X)$ by setting

Vertices: $g \in G$

Edges: for all $g \in G, x \in X$, $\ g \xrightarrow{x} gp(x)$

Preferred squares: for all $g \in G, x, y \in X$,

$$\begin{tikzcd} gp(x \triangleright y) \arrow[r, "x"] & gp(x)p(y) \\ g \arrow[u, "x \triangleright y"] \arrow[r, "x"'] & gp(x) \arrow[u, "y"'] \end{tikzcd}$$

Moreover, if we suppose that this augmented rack is pointed, then $\mathcal{T}_G(X)$ is a pointed corner trunk with identity 1.

Example A.2.10 (the n-cube)**.** We define for all $n \in \mathbb{N}$ a corner trunk, called *n-cube* and denoted $\square_n$, by setting

Vertices: subsets of $\{1, \ldots, n\}$

Edges: $V \xrightarrow{i} V \cup \{i\}$ where $i \notin V$

Preferred squares: for all $i < j$

$$\begin{tikzcd}
V \cup \{j\} \arrow[r, "i"] & V \cup \{i\} \cup \{j\} \\
V \arrow[u, "j"] \arrow[r, "i"] & V \cup \{i\} \arrow[u, "j"']
\end{tikzcd}$$

Another description of this trunk is

Vertices: n-uplets $(\epsilon_1, \ldots, \epsilon_n)$ where $\epsilon_i \in \{0, 1\}$

Edges: for all $i \in \{1, \ldots, n\}$, $(\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon_i, \epsilon_{i+1}, \ldots, \epsilon_n) \xrightarrow{i} (\epsilon_1, \ldots, \epsilon_{i-1}, \epsilon'_i, \epsilon_{i+1}, \ldots, \epsilon_n)$ where $\epsilon'_i = 1$.

Preferred squares: for all $i < j$

$$\begin{tikzcd}
(\epsilon_1, \ldots, \epsilon''_j, \ldots, \epsilon_n) \arrow[r, "i"] & (\epsilon_1, \ldots, \epsilon'_i, \ldots, \epsilon''_j, \ldots, \epsilon_n) \\
(\epsilon_1, \ldots, \epsilon_n) \arrow[u, "j"] \arrow[r, "i"] & (\epsilon_1, \ldots, \epsilon'_i, \ldots, \epsilon_n) \arrow[u, "j"']
\end{tikzcd}$$

Moreover, we define for all $n \in \mathbb{N}$ a pointed corner trunk $\square_n^+$ by deleting the hypothesis $i \notin V$ in the first definition of the vertices. Hence, the identities are $V \xrightarrow{id} V$.

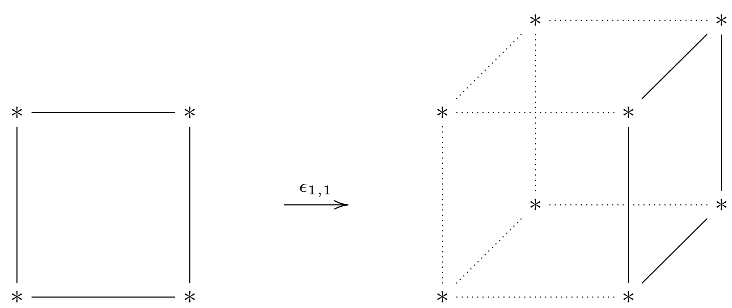
A.3 The category $\square^+$

Objects: We take as objects the trunks $\square_n^+$ defined above.
Morphisms: To define the morphisms in this category, we have to define two kinds of maps, the face maps and the degeneracies.

Face maps: a $(n-1)$-*face* is a map $\epsilon_{i,\alpha} : \square_n^+ \to \square_n^+$ given by

$$\epsilon_{i,\alpha}(\epsilon_1, \ldots, \epsilon_{n-1}) = (\epsilon_1, \ldots, \epsilon_{i-1}, \alpha, \epsilon_i, \ldots, \epsilon_{n-1})$$

A composition of such $(n-1)$-faces is called a *face map*.
Example: For $i = 1, \alpha = 1$, we can represent $\epsilon_{1,1}$ by



Degeneracies: a n-*degeneracy* is a map $\eta_i : \Box_n^+ \to \Box_{n-1}^+$ given by

$$\eta_i(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \dots, \epsilon_n)$$

A composition of such n-degeneracies is called a *degeneracy.*

Proposition A.3.1. *Face maps and degeneracies satisfy the following relations*

$$\epsilon_{i,\alpha}\epsilon_{j-1,\omega} = \epsilon_{j,\omega}\epsilon_{i,\alpha}, \quad 1 \leqslant i < j \leqslant n \text{ and } \alpha, \omega \in \{0,1\} \tag{A.1}$$

$$\eta_{j-1}\eta_i = \eta_i\eta_j, \quad i < j \tag{A.2}$$

$$\eta_j\epsilon_{i,\alpha} = \begin{cases} \epsilon_{i,\alpha}\eta_{j-1} & \text{if } i < j \\ \text{identity} & \text{if } i = j \\ \epsilon_{i-1,\alpha}\eta_j & \text{if } i > j \end{cases} \tag{A.3}$$

A.4 $\Box^+$-set

Definition A.4.1. *Let $\mathcal{C}$ be a category. A $\Box^+$**-object** in $\mathcal{C}$ is a functor $S : (\Box^+)^{op} \to \mathcal{C}$. A $\Box^+$**-map** between two $\Box^+$-objects in $\mathcal{C}$, S and S', is a natural transformation $\tau : S \Longrightarrow S'$.*

We can associate to any trunk a $\Box^+$-set called the *nerve.*

Definition A.4.2. *Let $\mathcal{T}$ be a trunk, the **nerve** of $\mathcal{T}$, denoted $N(\mathcal{T})$, is the $\Box^+$-set given by*

$$N(\mathcal{T})(\Box_n^+) = Hom(\Box_n^+, \mathcal{T})$$

$$N(\mathcal{T})(\lambda) = Hom(\Box_n^+, \mathcal{T}) \xrightarrow{\lambda^*} Hom(\Box_p^+, \mathcal{T}) \text{ for } \lambda : \Box_p^+ \to \Box_n^+$$

In the case where $\mathcal{C}$ is a category of modules, each $\Box^+$-module S determines a chain complex $\{C(S)_n, d_n\}_{n\geq 0}$ by setting

$$C(S)_n = S(\Box_n^+)$$

$$\partial_n = \sum_{k=1}^{n}(-1)^{k+1}(S(\epsilon_{k,0}) - S(\epsilon_{k,1}))$$

A.5 The nerve of a pointed augmented rack

Let $X \xrightarrow{p} G$ be a pointed augmented rack. We have seen in the Example A.2.9 that there is a pointed corner trunk $\mathcal{T}_G(X)$ associated to it. We will determine the nerve of $\mathcal{T}_G(X)$. This computation is facilitated by the following remark.

Remark A.5.1. Given edges $g_0 \xrightarrow{x_1} g_0p(x_1)$, $g_0 \xrightarrow{x_2} g_0p(x_2)$ and $g_0 \xrightarrow{x_3} g_0p(x_3)$ in $\mathcal{T}_G(X)$, then there exists a unique way to complete the diagram

$$\begin{tikzcd}
 & & * \arrow[rr, dotted] & & * \\
g_0p(x_3) \arrow[urr, dotted] \arrow[rrr, dotted] & & & * \arrow[ur, dotted] & \\
 & g_0p(x_2) \arrow[uur, dotted] \arrow[rrr, dotted] & & & * \arrow[uu, dotted] \\
g_0 \arrow[uu, "x_3"] \arrow[ur, "x_2"] \arrow[rrr, "x_1"] & & & g_0p(x_1) \arrow[uu, dotted] \arrow[ur, dotted] &
\end{tikzcd}$$

into a diagram of preferred squares. An equivalent statement is to say that we have the identity

$$c_{x_1}^{-1} \circ c_{x_2}^{-1} = c_{c_{x_1}^{-1}(x_2)}^{-1} \circ c_{x_1}^{-1}$$

Proposition A.5.2. *Let $X \xrightarrow{p} G$ be a pointed corner trunk. We have a bijection*

$$N(\mathcal{T}_G(X))(\square_n^+) \overset{t_\square^n}{\rightarrow} G \times X^n, \forall n \in \mathbb{N}$$

and

$$\begin{aligned}
N(\mathcal{T}_G(X))(\epsilon_{i,0})(g_0, x_1, \dots, x_n) &= (g_0, x_1, \dots, x_i \triangleright x_{i+1}, \dots, x_i \triangleright x_n) \\
N(\mathcal{T}_G(X))(\epsilon_{i,1})(g_0, x_1 \dots, x_n) &= \big(g_0 p(x_1 \triangleright \dots \triangleright x_i), x_1, \dots, \widehat{x_i}, \dots, x_n\big) \\
N(\mathcal{T}_G(X))(\eta_i)(g_0, x_1, \dots, x_n) &= (g_0, x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n)
\end{aligned}$$

Proof : Let $F \in N(\mathcal{T}_G(X))(\square_n^+)$, we define an element $(g_0, x_1, \dots, x_n) \in G \times X^n$ by setting $g_0 = F(\emptyset)$ and

$$x_k = F(\{1, \dots, k-1\}) \overset{\{i\}}{\rightarrow} F(\{1, \dots, k\})$$

Conversely, let $(g_0, x_1, \dots, x_n)$, we define an element F on the subsets $\{1, \dots, k\}$ of $\{1, \dots, n\}$ by setting

$$F(\{1, \dots, k\} = g_0 p(x_1) \dots p(x_k)$$

these data allow us to define F on the subsets $\{k\}$. Indeed, we have the following diagram of preferred square in $\square_n^+$

$$\begin{tikzcd}
\{k\} \arrow[r, "1"] & \{1,k\} \arrow[r, "2"] & \{1,2,k\} \arrow[r] & \cdots \arrow[r] & \{1,\dots,k-2,k\} \arrow[r, "k-1"] & \{1,\dots,k\} \\
\emptyset \arrow[u, "k"] \arrow[r, "1"] & \{1\} \arrow[u, "k"] \arrow[r, "2"] & \{1,2\} \arrow[u, "k"] \arrow[r] & \cdots \arrow[r] & \{1,\dots,k-2\} \arrow[u, "k"] \arrow[r, "k-1"] & \{1,\dots,k-1\} \arrow[u, "k"]
\end{tikzcd}$$

Hence, by the corner axiom $(C1)$, if we want F to be a trunk map, we have a unique choice for $F(\{k\})$. Thus, F is determined on $\{k\}$ for all $1 \leq k \leq n$.
Now we can construct F by induction on the cardinality of the subsets of $\{1, \dots, n\}$. Suppose

that F is known on the subset of $\{1,\dots,n\}$ of cardinal lower than l. Let A be a subset of $\{1,\dots,n\}$ of cardinal l, $i \notin A$ and $B \subset A$ of cardinal $l-1$. The square

$$\begin{tikzcd}
A \arrow[r, "\{i\}"] & A \cup \{i\} \\
B \arrow[u] \arrow[r, "\{i\}"] & B \cup \{i\} \arrow[u]
\end{tikzcd}$$

is preferred in $\square_n^+$. We want that F sends preferred squares to preferred squares, hence by using the remark above, we have a unique choice to define the edge $F(A) \to F(A \cup \{i\})$. To show that this is well defined, we have to show that our construction does not depend on the chosen subset B. Suppose that $A = B_1 \cup \{j\} = B_2 \cup \{k\}$ and let $B = B_1 \cap B_2$ so that $A = B \cup \{j,k\}$ and suppose, without loss of generality, that $j < k$. We have the following diagram of preferred squares in $\square_n^+$

$$\begin{tikzcd}
& B \cup \{i,k\} \arrow[rr] & & A \cup \{i\} \\
B \cup \{k\} \arrow[ur] \arrow[rr] & & A \arrow[ur] & \\
& B \cup \{i\} \arrow[uu, dotted] \arrow[rr, dotted] & & B \cup \{i,j\} \arrow[uu] \\
B \arrow[uu] \arrow[ur, dotted] \arrow[rr] & & B \cup \{j\} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

By induction, F is defined on the three edges originating at B. Hence if we want F to be a trunk map, by using the remark above $F(A) \to F(A \cup \{i\})$ is well defined.

Let $(g_0, x_1, \dots, x_n) \in G \times X^n$ and let F be the element of $N(\mathcal{T}_G(X))$ associated to this $(n+1)$-tuple. We have for all $1 \le k \le n$, $\epsilon_{i,0}(\{1,\dots,k\}) = \{1,\dots,i-1,i+1,\dots,k+1\}$. We have the following diagram of preferred squares in $\square_n^+$

$$\begin{tikzcd}
\{1,\dots,i-1\} \arrow[r] \arrow[d] & \{1,\dots,i-1,i+1\} \arrow[r, dash, dotted] \arrow[d] & \{1,\dots,i-1,i+1,\dots,k\} \arrow[r] \arrow[d] & \{1,\dots,i-1,i+1,\dots k+1\} \arrow[d] \\
\{1,\dots,i\} \arrow[r] & \{1,\dots,i+1\} \arrow[r, dash, dotted] & \{1,\dots,k\} \arrow[r] & \{1,\dots,k+1\}
\end{tikzcd}$$

F is a trunk map, thus it sends this diagram to the following diagram of preferred squares in $\mathcal{T}_G(X)$

$$\begin{tikzcd}
g_0 p(x_1)\dots p(x_{i-1}) \arrow[r] \arrow[d, "x_i"] & F(\epsilon_{i,0}(\{1,\dots,i\})) \arrow[r, dash, dotted] \arrow[d, "x_i"] & F(\epsilon_{i,0}(\{1,\dots,k-1\})) \arrow[r] \arrow[d, "x_i"] & F(\epsilon_{i,0}(\{1,\dots,k\})) \arrow[d, "x_i"] \\
g_0 p(x_1)\dots p(x_i) \arrow[r, "x_{i+1}"] & g_0 p(x_1)\dots p(x_{i+1}) \arrow[r, dash, dotted] & g_0 p(x_1)\dots p(x_k) \arrow[r, "x_k"] & g_0 p(x_1)\dots p(x_{k+1})
\end{tikzcd}$$

Thus $F(\epsilon_{i,0}(\{1,\dots,k-1\})) \to F(\epsilon_{i,0}(\{1,\dots,k\}))$ is equal to $x_i \triangleright x_k$ if $i \le k$ and x_i if $i > k$. Hence

$$N(\mathcal{T}_G(X))(\epsilon_{i,0})(g_0, x_1, \dots, x_n) = (g_0, x_1, \dots, x_i \triangleright x_{i+1}, \dots, x_i \triangleright x_n)$$

Moreover, we have for all $1 \leq k \leq n$, $\epsilon_{i,1}(\{1,\dots,k\}) = \{1,\dots,k+1\}$ if $i \leq k$ and $\epsilon_{i,1}(\{1,\dots,k\}) = \{1,\dots,k,i\}$ if $i > k$. Thus $F(\epsilon_{i,1}(\{1,\dots,k-1\})) \to F(\epsilon_{i,1}(\{1,\dots,k\}))$ is equal to x_{k+1} if $i \leq k$. For $i > k$, we have the following preferred square

$$\begin{tikzcd}
\{1,\dots,k-1,i\} \arrow[r] & \{1,\dots,k,i\} \\
\{1,\dots,k-1\} \arrow[r] \arrow[u] & \{1,\dots,k\} \arrow[u]
\end{tikzcd}$$

Thus $F(\epsilon_{i,1}(\{1,\dots,k-1\})) \to F(\epsilon_{i,1}(\{1,\dots,k\}) = F(\{1,\dots,k-1\}) \to F(\{1,\dots,k\}) = x_k$. It remains to determine $F(\epsilon_{i,1}(\emptyset))$. We have $\epsilon_{i,1}(\emptyset) = \{i\}$ and the following diagram is a diagram of preferred squares in $\mathcal{T}_G(X)$.

$$\begin{tikzcd}
F(\{i\}) \arrow[r, "x_1"] & F(\{1,i\}) \arrow[r, "x_2"] & \cdots \arrow[r] & F(\{1,\dots,i-2,i\}) \arrow[r, "x_{i-1}"] & F(\{1,\dots,i\}) \\
F(\emptyset) \arrow[r, "x_1"] \arrow[u, "x_1 \triangleright \dots \triangleright x_i"] & F(\{1\}) \arrow[r, "x_2"] \arrow[u, "x_2 \triangleright \dots \triangleright x_i"] & \cdots \arrow[r] & F(\{1,\dots,i-2\}) \arrow[r, "x_{i-1}"] \arrow[u, "x_{i-1} \triangleright x_i"] & F(\{1,\dots,i-1\}) \arrow[u, "x_i"']
\end{tikzcd}$$

Thus by the corner axiom $(C1)$, $F(\epsilon_{i,1}(\emptyset)) = g_0 p(x_1 \triangleright \dots \triangleright x_i)$ and

$$N(\mathcal{T}_G(X))(\epsilon_{i,1})(g_0, x_1 \dots, x_n) = \big(g_0 p(x_1 \triangleright \dots \triangleright x_i), x_1, \dots, \widehat{x_i}, \dots, x_n\big)$$

Furthermore, we have for all $1 \leq k \leq n$, $\eta_i(\{1,\dots,k\}) = \{1,\dots,k-1\}$ if $i \leq k$ and $\eta_i(\{1,\dots,k\}) = \{1,\dots,k\}$ if $i > k$. Thus $F(\eta_i(\{1,\dots,k-1\})) \to F(\eta_i(\{1,\dots,k\}))$ is equal to x_{k-1} if $i > k$ and is equal to x_k if $i < k$. For $i = k$, we have $F(\eta_k(\{1,\dots,k-1\})) \to F(\eta_k(\{1,\dots,k\})) = F(\{1,\dots,k-1\}) \to F(\{1,\dots,k-1\}) = 1$. Hence

$$N(\mathcal{T}_G(X))(\eta_i)(g_0, x_1, \dots, x_n) = (g_0, x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n).$$

□

We denote $N(\mathcal{T}_G(X))(\square_n^+)$ by $\Gamma_G^n(X)$, $N(\mathcal{T}_G(X))(\epsilon_{i,\alpha})$ by $\partial_{i,\alpha}$ and $N(\mathcal{T}_G(X))(\eta_i)$ by σ_i.

Proposition A.5.3. *For all $n \in \mathbb{N}$, we define an action of G on $\Gamma_G(X)$ by setting*

$$g.(g_0, x_1, \dots, x_n) = (gg_0, x_1, \dots, x_n).$$

Moreover, the action preserves the maps $\partial_{i,\alpha}$ and σ_i, that is for all $\gamma \in \Gamma_G(X)$

$$g.\partial_{i,\alpha}(\gamma) = \partial_{i,\alpha}(g.\gamma),$$

and

$$g.\sigma_i(\gamma) = \sigma_i(g.\gamma).$$

Let A be a G-module. We denote by $E^n(\Gamma_G(X), A)$ the set of functions from $\Gamma_G^n(X)$ to A which are equivariant for the action of G, and such that $f \circ \sigma_i = 0$, for all $1 \leq i \leq n$. We define a cochain complex $\{E^n(\Gamma_G(X), A), d_\Gamma^n\}_{n \in \mathbb{N}}$ by setting

$$d_\Gamma^n f = \sum_{i=1}^{n} (-1)^{i+1} (d_\Gamma^{i,1} f - d_\Gamma^{i,0} f),$$

where $d_\Gamma^{i,\alpha} f = f \circ \partial_{i,\alpha}$.

Lemma A.5.4. $d_\Gamma^{n+1} \circ d_\Gamma^n = 0$.

Proof : This is clear using the identities (A.1).

□

A.6 Relation with pointed rack cohomology

Let X be a pointed rack. We have seen in Example 2.1.24 that there is a canonical pointed augmented rack associated to X, it is $X \xrightarrow{\mu} As_p(X)$. Recall that on an $As_p(X)$-module A, we can define the structure of symmetric X-pointed module. The following proposition establishes an isomorphism between the cochain complexes $\{E^n(\Gamma_{As_p(X)}(X), A), d^n_\Gamma\}_{n\in\mathbb{N}}$ and $\{CR^n_p(X, A^s), d^n_R\}_{n\in\mathbb{N}}$.

Proposition A.6.1. *Let X be a pointed rack and let A be an $As_p(X)$-module. We have an isomorphism of cochain complexes*

$$E^n(\Gamma_{As_p(X)}(X), A) \xrightarrow{\nu^n} CR^n_p(X, A^s),$$

given by

$$\nu^n(f)(x_1, \ldots, x_n) = f([1], x_1, \ldots, x_n),$$

with inverse given by

$$(\nu^n)^{-1}(f)([w], x_1, \ldots, x_n) = [w].f(x_1, \ldots, x_n).$$

Proof : Let $f \in E^n(\Gamma_{As_p(X)}(X), A)$ and $(x_1, \ldots, x_n) \in X^n$. We have

$$\begin{aligned}
d^n_R\nu^n(f)(x_1, \ldots, x_{n+1}) &= \sum_{i=1}^{n}(-1)^{n+1}\big(\phi_{x_1\rhd\ldots,\rhd x_i}(\nu^n(f)(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})\\
&\qquad - \nu^n(f)(x_1, \ldots, x_{i-1}, x_i\rhd x_{i+1}, \ldots, x_i\rhd x_{n+1})\big)\\
&= \sum_{i=1}^{n}(-1)^{n+1}\big(\phi_{x_1\rhd\ldots,\rhd x_i}(f([1], x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})\\
&\qquad - f([1], x_1, \ldots, x_{i-1}, x_i\rhd x_{i+1}, \ldots, x_i\rhd x_{n+1})\big)\\
&= \sum_{i=1}^{n}(-1)^{n+1}\big((f([x_1\rhd\ldots,\rhd x_i], x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})\\
&\qquad - f([1], x_1, \ldots, x_{i-1}, x_i\rhd x_{i+1}, \ldots, x_i\rhd x_{n+1})\big),
\end{aligned}$$

and

$$\begin{aligned}
\nu^{n+1}(d^n_\Gamma f)(x_1, \ldots, x_{n+1}) &= d^n_\Gamma f([1], x_1, \ldots, x_{n+1})\\
&= \sum_{i=1}^{n}(-1)^{i+1}(d^{i,1}_\Gamma f - d^{i,0}_\Gamma f)([1], x_1, \ldots, x_{n+1})\\
&= \sum_{i=1}^{n}(-1)^{i+1}(f\circ\epsilon_{i,1} - f\circ\epsilon_{0,\alpha})([1], x_1, \ldots, x_{n+1})\\
&= \sum_{i=1}^{n}(-1)^{i+1}\big(f([x_1\rhd\ldots,\rhd x_i], x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})\\
&\qquad - f([1], x_1, \ldots, x_{i-1}, x_i\rhd x_{i+1}, \ldots, x_i\rhd x_{n+1})\big).
\end{aligned}$$

Thus $\{\nu^n\}_{n\in\mathbb{N}}$ is an isomorphism of cochain complexes.

$\square$

A.7 Relation with group cohomology

Let G be a group, Proposition 2.3.24 establishes that there is a morphism from the group cohomology of G to the pointed rack cohomology of $Conj(G)$. Here we give a proof of this theorem using trunk theory.

The $\square^+$-complex associated to a group

We can see a group as the pointed augmented rack $Conj(G) \xrightarrow{id} G$, thus if we have a G-module A, we can talk about $\{E^n(\Gamma_G(Conj(G)), A), d^n_\Gamma\}_{n\in\mathbb{N}}$.

Proposition A.7.1. *We have an isomorphism of cochain complexes*

$$E^n(\Gamma_G(G), A) \xrightarrow{t^n} E^n(\Gamma_{As(Conj(G))}(Conj(G)), A),$$

given by

$$t^n(f)([w], g_1, \ldots, g_n) = [w].f(1, g_1, \ldots, g_n),$$

with inverse given by

$$(t^n)^{-1}(f)(g_0, g_1, \ldots, g_n) = f([g_0], g_1, \ldots, g_n).$$

Hence, by combining Proposition A.6.1 and Proposition A.7.1, we have the following result

Proposition A.7.2. *We have an isomorphism of cochain complexes*

$$E^n(\Gamma_G(G), A) \xrightarrow{\theta^n} CR^n_p(G, A^s),$$

given by

$$\theta^n(f)(g_1, \ldots, g_n) = f(1, g_1, \ldots, g_n),$$

with inverse given by

$$(\theta^n)^{-1}(f)(g_0, \ldots, g_n) = g_0.f(g_1, \ldots, g_n).$$

The Δ-complex associated to a group

In the introduction we have associated a category to a group G. We can use this construction to associate to G a simplicial complex with an action of the group G. By taking the equivariant map from this simplical complex to a G-module A, we construct a cochain complex and we can show that there exists an isomorphism between this complex and the complex defining the group cohomology of G.

Let G be a group. Recall the description of the category $\mathcal{C}_G(G)$

Objects: $g \in G$

Morphisms: $g \xrightarrow{h} gh$

Composition: $g \xrightarrow{h} gh \circ gh \xrightarrow{k} ghk = g \xrightarrow{ghk} ghk$

Identity: $g \xrightarrow{1} g$

The nerve of the category $\mathcal{C}_G(G)$ is the simplicial set $N(\mathcal{C}_G(G))$ defined by

$$N(\mathcal{C}_G(G))(\Delta_n) = Hom(\Delta_n, \mathcal{C}_G(G))$$
$$N(\mathcal{C}_G(G))(\lambda) = Hom(\Delta_n, \mathcal{C}_G(G)) \xrightarrow{\lambda^*} Hom(\Delta_p, \mathcal{C}_G(G)) \text{ for } \lambda : \Delta_p \to \Delta_n$$

Proposition A.7.3. *Let G be a group. We have a bijection*

$$N(\mathcal{C}_G(G))(\Delta_n) \xrightarrow{t^n_\Delta} G^{n+1}, \forall n \in \mathbb{N},$$

and under this isomorphism

$$N(\mathcal{C}_G(G))(\epsilon_i)(g_0, \dots, g_n) = \begin{cases} (g_1, \dots, g_n) & \text{if } i = 0 \\ (g_0, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 1 \leq i \leq n-1 \\ (g_0, \dots, g_{n-1}) & \text{if } i = n \end{cases}$$
$$N(\mathcal{C}_G(G))(\eta_i)(g_0, \dots, g_n) = (g_0, \dots, 1, \dots, g_n)$$

Proof : For all $F \in N(\mathcal{C}_G(G))$, we define $t^n_\Delta(F) = (g_0, g_1, \dots, g_n)$ by setting $g_0 = F(0)$ and

$$g_i = F(i-1) \to F(i).$$

Conversely, given a $(n+1)$-tuple in G^{n+1}, we define an element $((t^n_\Delta)^{-1})(g_0, \dots, g_n)$ in $N(\mathcal{C}_G(G))$ by setting

$$((t^n_\Delta)^{-1})(g_0, \dots, g_n))(i) = g_0 . \dots . g_i.$$

Hence $N(\mathcal{C}_G(G))$ and G^{n+1} are in bijection.
Moreover, let $(g_0, \dots, g_n) \in G^{n+1}$, we have

$$\begin{aligned}\big(N(\mathcal{C}_G(G)\big)(\epsilon_0)\big((t^n_\Delta)^{-1})(g_0, \dots, g_n)\big)(i) &= ((t^n_\Delta)^{-1})(g_0, \dots, g_n))(\epsilon_0(i)) \\ &= ((t^n_\Delta)^{-1})(g_0, \dots, g_n))(i+1).\end{aligned}$$

Thus $\big(t^{n-1}_\Delta \circ N(\mathcal{C}_G(G))(\epsilon_0) \circ (t^n_\Delta)^{-1}\big)(g_0, \dots, g_n) = (g_1, \dots, g_n)$.
We have

$$\begin{aligned}\big(N(\mathcal{C}_G(G)\big)(\epsilon_n)\big((t^n_\Delta)^{-1})(g_0, \dots, g_n)\big)(i) &= ((t^n_\Delta)^{-1})(g_0, \dots, g_n))(\epsilon_n(i)) \\ &= ((t^n_\Delta)^{-1})(g_0, \dots, g_n))(i).\end{aligned}$$

Thus $\big(t^{n-1}_\Delta \circ N(\mathcal{C}_G(G))(\epsilon_n) \circ (t^n_\Delta)^{-1}\big)(g_0, \dots, g_n) = (g_0, \dots, g_{n-1})$.
Furthermore, for $1 \leq k \leq n-1$, we have

$$\big(N(\mathcal{C}_G(G)\big)(\epsilon_k)\big((t^n_\Delta)^{-1})(g_0, \dots, g_n)\big)(i) = \begin{cases} ((t^n_\Delta)^{-1})(g_0, \dots, g_n)\big)(i) & \text{if } i < k \\ ((t^n_\Delta)^{-1})(g_0, \dots, g_n)\big)(i+1) & \text{if } i \geq k \end{cases}$$

Thus $\big(t^{n-1}_\Delta \circ N(\mathcal{C}_G(G))(\epsilon_k) \circ (t^n_\Delta)^{-1}\big)(g_0, \dots, g_n) = (g_0, \dots, g_k g_{k+1}, \dots, g_{n-1})$.
To finish, let $1 \leq k \leq n$, we have

$$\big(N(\mathcal{C}_G(G)\big)(\eta_k)\big((t^n_\Delta)^{-1})(g_0, \dots, g_n)\big)(i) = \begin{cases} ((t^n_\Delta)^{-1})(g_0, \dots, g_n)\big)(i) & \text{if } i \leq k \\ ((t^n_\Delta)^{-1})(g_0, \dots, g_n)\big)(i-1) & \text{if } i > k \end{cases}$$

Thus $\big(t^{n-1}_\Delta \circ N(\mathcal{C}_G(G))(\eta_k) \circ (t^n_\Delta)^{-1}\big)(g_0, \dots, g_n) = (g_0, \dots, 1, \dots, g_{n-1})$.

$\square$

We denote $N(\mathcal{C}_G(G))(\Delta_n)$ by $\Sigma^n_G(G)$, $N(\mathcal{C}_G(G))(\epsilon_i)$ by ∂_i and $N(\mathcal{C}_G(G))(\eta_i)$ by σ_i.

Proposition A.7.4. *For all $n \in \mathbb{N}$, we define an action of G on $\Sigma_G(G)$ by setting*

$$g.(g_0, g_1, \ldots, g_n) = (gg_0, g_1, \ldots, g_n).$$

Moreover, the action preserves the maps ∂_i and σ_i, that is, for all $\gamma \in \Sigma_G(G)$

$$g.\partial_i(\gamma) = \partial_i(g.\gamma),$$

and

$$g.\sigma_i(\gamma) = \sigma_i(g.\gamma).$$

Let A be a G-module. We denote by $E^n(\Sigma_G(G), A)$ the set of functions from $\Sigma^n_G(G)$ to A which are equivariant for the action of G, and such that $f \circ \sigma_i = 0$, for all $1 \leq i \leq n$. We define a cochain complex $\{E^n(\Sigma_G(G), A), d^n_\Sigma\}_{n\in\mathbb{N}}$ by setting

$$d^n_\Sigma f = \sum_{i=0}^{n+1} (-1)^{i+1} d^i_\Sigma f,$$

where $d^i_\Sigma f = f \circ \partial_i$.

Lemma A.7.5. $d^{n+1}_\Sigma \circ d^n_\Sigma = 0$.

Proof : This is clear because $\{E^n(\Sigma_G(G), A)\}_{n\in\mathbb{N}}$ is a cosimplicial module.

$\square$

Proposition A.7.6. *Let G be a group and let A be a G-module. We have an isomorphism of cochain complexes*

$$E^n(\Sigma_G(G), A) \xrightarrow{\zeta^n} C^n(G, A),$$

given by

$$\zeta^n(f)(g_1, \ldots, g_n) = f(1, g_1, \ldots, g_n),$$

with inverse given by

$$(\zeta^n)^{-1}(f)(g_0, g_1, \ldots, g_n) = g_0.f(g_1, \ldots, g_n).$$

Proof : Let $f \in E^n(\Sigma_G(G), A)$ and $(g_1, \ldots, g_{n+1}) \in G^{n+1}$. We have

$$\begin{aligned}
d^n\zeta^n(f)(g_1, \ldots, g_{n+1}) &= g_1.\zeta^n(f)(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \zeta(f)(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\
&\quad + (-1)^{n+1}\zeta^n(f)(g_1, \ldots, g_n) \\
&= g_1.f(1, g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(1, g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(1, g_1, \ldots, g_n) \\
&= f(g_1, g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(1, g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(1, g_1, \ldots, g_n),
\end{aligned}$$

and

$$\begin{aligned}\zeta^{n+1}(d^n_\Sigma f)(g_1,\dots,g_{n+1}) &= d^n_\Sigma f(1,g_1,\dots,g_{n+1})\\ &= \sum_{i=0}^{n+1}(-1)^i d^i_\Sigma f(1,g_1,\dots,g_{n+1})\\ &= f(g_1,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(1,\dots,g_ig_{i+1},\dots,g_{n+1}) + (-1)^{n+1} f(1,g_1,\dots,g_n).\end{aligned}$$

Hence $\{\zeta^n\}_{n\in\mathbb{N}}$ is an isomorphism of cochain complexes.

□

A morphism from $E^n(\Sigma_G(G),A)$ to $E^n(\Gamma_G(G),A)$

In this part we construct a morphism of cochain complexes from $\{E^n(\Sigma_G(G),A), d^n_\Sigma\}_{n\in\mathbb{N}}$ to $\{E^n(\Gamma_G(G),A), d^n_\Gamma\}_{n\in\mathbb{N}}$, and we show that under the isomorphisms given in Proposition A.7.2 and Proposition A.7.6, this morphism is equal to $\{\Delta^n\}_{n\in\mathbb{N}}$ defined in Proposition 2.3.24.

To construct this morphism, we define for all $n\in\mathbb{N}$ a finite family of maps ($n!$ exactly) from $\Gamma^n_G(G)$ to $\Sigma^n_G(G)$ which commute with $\partial_{i,\alpha}$ and ∂_i. By definition, these complexes are the sets of pointed trunk-maps from $\Box^+$ to $\mathcal{T}_G(G)$ and the set of functors from Δ to $\mathcal{C}_G(G)$. Hence, we want to define for all $n\in\mathbb{N}$, a finite family of maps

$$Hom_{Trunk}(\Box^+_n, \mathcal{T}_G(G)) \longrightarrow Hom_{Cat}(\Delta_n, C_G(G))$$

Every map in this family is the composition of three maps

$$\begin{tikzcd} Hom_{Trunk}(\Box^+_n, \mathcal{T}_G(G)) \arrow[r, dotted] \arrow[d] & Hom_{Cat}(\Delta_n, C_G(G)) \\ Hom_{Trunk}(\Box^+_n, Trunk(C_G(G))) \arrow[r] & Hom_{Cat}(Cat(\Box^+_n), C_G(G)) \arrow[u] \end{tikzcd}$$

where the map

$$Hom_{Trunk}(\Box^+_n, \mathcal{T}_G(G)) \to Hom_{Trunk}(\Box^+_n, Trunk(\mathcal{C}_G(G))),$$

is induced by the canonical map $\mathcal{T}_G(G)\to Trunk(\mathcal{C}_G(G))$. The map

$$Hom_{Trunk}(\Box^+_n, Trunk(C_G(G))) \to Hom_{Cat}(Cat(\Box^+_n), C_G(G))$$

is the bijection defined using the adjointness between the functors $Trunk$ and Cat, and the map

$$Hom_{Cat}(Cat(\Box^+_n), C_G(G)) \to Hom_{Cat}(\Delta_n, C_G(G))$$

is induced by an injection of Δ_n in $Cat(\Box^+_n)$. It is the last application which gives us $n!$ choices of maps.

We want to define a map from $Hom_{Trunk}(\Box^+_n, \mathcal{T}_G(G))$ to $Hom_{Trunk}(\Box^+_n, Trunk(\mathcal{C}_G(G)))$. We have a canonical pointed trunk-map from $\mathcal{T}_G(G)$ to $Trunk(\mathcal{C}_G(G))$ given by the identity on the vertices and on the edges. Hence, the composition by this pointed trunk-map gives the wanted map.

To define the map from $Hom_{Trunk}(\square_n^+, Trunk(C_G(G)))$ to $Hom_{Cat}(Cat(\square^n), C_G(G))$, we use the adjointness property between the functors $Trunk$ and Cat. It gives us a bijective map from $Hom_{Trunk}(\square_n^+, Trunk(C_G(G)))$ to $Hom_{Cat}(Cat(\square^n), C_G(G))$.

It remains to define the maps from $Hom_{Cat}(Cat(\square_n^+), C_G(G))$ to $Hom_{Cat}(\Delta_n, C_G(G))$. For this, we use canonical injections from Δ_n to $Cat(\square_n^+)$. The injections that we consider are the followings; we define them recursively by

$$0 \mapsto (0, \dots, 0)$$

and

$$k \mapsto (\alpha_1^k, \dots, \alpha_n^k)$$

such that $\alpha_l^k = 1$ if $\alpha_l^{k-1} = 1$, and such that there exists a unique $i \in \{1, \dots, n\}$ with $\alpha_i^{k-1} \neq 1$ and $\alpha_i^k = 1$. There are exactly $n!$ different maps defined in this way. Indeed, if the image of $[k-1]$ is fixed, we have $n-k$ choices for $[k]$. Hence, we have $n!$ choices. If we use the description of $Cat(\square_n^+)$ in term of subsets of $\{1, \dots, n\}$, these functors are defined by

$$0 \mapsto \emptyset$$

and

$$k \mapsto V_k = V_{k-1} \cup \{i\}$$

where $i \notin V_{k-1}$. It is better to have explicit formulas for these maps. Let $(i_1, \dots, i_n)$ be a n-tuple in $\{1, \dots, n\}$, we denote by $\lambda_{i_1,\dots,i_n}$ the functor

$$k \mapsto (\alpha_1^k, \dots, \alpha_n^k)$$

where $\alpha_i^k = 1$ if and only if $i \in (i_1, \dots, i_k)$. Another way to define them is to index by the elements of the symmetric group $\mathfrak{S}_n$. Let $\sigma \in \mathfrak{S}_n$, we denote by λ_σ the functor

$$k \mapsto (\alpha_1^k, \dots, \alpha_n^k) \tag{A.4}$$

where $\alpha_i^k = 1$ if and only if $i \in (\sigma(1), \dots, \sigma(k))$. We will use this formulation in the sequel.

Composing these four maps, for all $n \in \mathbb{N}$ we obtain a family of maps from $\Gamma_G^n(G) \xrightarrow{\Lambda_\sigma^n} \Sigma_G^n(G)$ indexed on the symmetric group $\mathfrak{S}_n$ defined by

$$\Lambda_\sigma^n(F) = F \circ \lambda_\sigma^n.$$

Using this family, we define for all $n \in \mathbb{N}$ a map $E^n(\Sigma_G(G), A) \xrightarrow{\Lambda^n} E^n(\Gamma_G(G), A)$ by setting

$$\Lambda^n(f) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)} f \circ \Lambda_\sigma^n. \tag{A.5}$$

Proposition A.7.7. *Let G be a group and let A be a G-module, then*

$$\{E^n(\Sigma_G(G), A), d_\Sigma^n\}_{n \in \mathbb{N}} \xrightarrow{\{\Lambda^n\}_{n \in \mathbb{N}}} \{E^n(\Gamma_G(G), A), d_\Gamma^n\}_{n \in \mathbb{N}}$$

is a morphism of cochain complexes.

To prove this proposition we need a lemma linking $\lambda_\sigma, \epsilon_{i,\alpha}$ and ϵ_i.

Lemma A.7.8. *Let $\sigma \in \mathfrak{S}_n$ and $i \in \{1, \dots, n\}$. We have the following relations*

$$\lambda_\sigma \circ \epsilon_k = \lambda_{\sigma \circ (k\, k+1)} \circ \epsilon_k \text{ for } 1 \le k \le n-1 \tag{A.6}$$
$$\epsilon_{i,0} \circ \lambda_\sigma = \lambda_{\tau^0_{\sigma,i}} \circ \epsilon_n \tag{A.7}$$
$$\epsilon_{i,1} \circ \lambda_\sigma = \lambda_{\tau^1_{\sigma,i}} \circ \epsilon_0 \tag{A.8}$$

where $\tau^0_{\sigma,i}$ and $\tau^1_{\sigma,i}$ are defined by

$$\tau^0_{\sigma,i}(k) = \begin{cases} \sigma(k) & \text{if } \sigma(k) < i \\ \sigma(k)+1 & \text{if } i \le \sigma(k) \le n-1 \end{cases}$$
$$\tau^0_{\sigma,i}(n) \neq i$$

and

$$\tau^1_{\sigma,i}(k) = \begin{cases} \sigma(k-1) & \text{if } \sigma(k-1) < i \\ \sigma(k-1)+1 & \text{if } i \le \sigma(k-1) \end{cases}$$
$$\tau^1_{\sigma,i}(1) = i$$

Proof : Let $\sigma \in \mathfrak{S}_n$, we show first the relation $\lambda_\sigma \circ \epsilon_k = \lambda_{\sigma\circ(k\, k+1)} \circ \epsilon_k$ for $1 \le k \le n-1$. Let $1 \le k \le n-1$ and $l \in [n]$. We have

$$(\lambda_{\sigma\circ(k\, k+1)} \circ \epsilon_k)(l) = \begin{cases} \lambda_{\sigma\circ(k\, k+1)}(l) & \text{if } l < k \\ \lambda_{\sigma\circ(k\, k+1)}(l+1) & \text{if } k \le l \end{cases}$$

If $l < k$, then $\lambda_{\sigma\circ(k\, k+1)}(l) = (\alpha^l_1, \dots, \alpha^l_n)$ with $\alpha^l_i = 1$ if and only if $i \in \{(\sigma\circ(k\ \ k+1))(1), \dots, (\sigma\circ(k\ \ k+1))(l)\} = \{\sigma(1), \dots, \sigma(l)\}$. Thus, $\lambda_\sigma(l) = \lambda_{\sigma\circ(k\, k+1)}(l)$.
If $l = k$, then $\lambda_{\sigma\circ(k\, k+1)}(k+1) = (\alpha^{k+1}_1, \dots, \alpha^{k+1}_n)$ with $\alpha^{k+1}_i = 1$ if and only if $i \in \{(\sigma\circ(k\ \ k+1))(1), \dots, (\sigma\circ(k\ \ k+1))(k+1)\} = \{\sigma(1), \dots, \sigma(k+1)\}$. Thus, $\lambda_\sigma(k+1) = \lambda_{\sigma\circ(k\, k+1)}(k+1)$.
If $l > k$, then $\lambda_{\sigma\circ(k\, k+1)}(l+1) = (\alpha^{l+1}_1, \dots, \alpha^{l+1}_n)$ with $\alpha^{l+1}_i = 1$ if and only if $i \in \{(\sigma\circ(k\ \ k+1))(1), \dots, (\sigma\circ(k\ \ k+1))(l+1)\} = \{\sigma(1), \dots, \sigma(l)\}$. Thus, $\lambda_\sigma(l+1) = \lambda_{\sigma\circ(k\, k+1)}(l+1)$.

Let $k \in \{0, \dots, n\}$, we want to show that $(\epsilon_{i,1} \circ \lambda_\sigma)(k) = (\lambda_{\tau^1_{\sigma,i}} \circ \epsilon_0)(k)$. We have

$$(\epsilon_{i,1} \circ \lambda_\sigma)(k) = \epsilon_{i,1}(\alpha^k_1, \dots, \alpha^k_n) = (\gamma_1, \dots, \gamma_{n+1})$$

where

$$\gamma_j = \begin{cases} \alpha^k_j & \text{if } j < i \\ 1 & \text{if } i = j \\ \alpha^k_{j-1} & \text{if } j > i \end{cases}$$

and

$$(\lambda_{\tau^1_{\sigma,i}} \circ \epsilon_0)(k) = \lambda_{\tau^1_{\sigma,i}}(k+1) = (\beta^{k+1}_1, \dots, \beta^{k+1}_{n+1})$$

where $\beta^{k+1}_j = 1$ if and only if $j \in \{\tau^1_{\sigma,i}(1), \dots, \tau^1_{\sigma,i}(k+1)\}$. Then $\beta^{k+1}_j = 1$ if and only if there exists $l \in \{1, \dots, k+1\}$ such that $j = (\tau^1_{\sigma,i})(l)$.
If $j < i$, then $\beta^{k+1}_j = 1$ if and only if there exists $l \in \{2, \dots, k+1\}$ such that $j = \sigma(l-1)$. That is, $\beta^{k+1}_j = 1$ if and only if there exists $l \in \{1, \dots, k\}$ such that $j = \sigma(l)$.
If $j = i$, then $\beta^{k+1}_i = 1$ because $\tau^1_{\sigma,i}(1) = i$.
If $j > i$, then $\beta^{k+1}_j = 1$ if and only if there exists $l \in \{2, \dots, k+1\}$ such that $j = \sigma(l-1)+1$.

That is, $\beta_j^{k+1} = 1$ if and only if there exists $l \in \{1, \dots, k\}$ such that $j = \sigma(l) + 1$.
Hence for all j, $\gamma_j = \beta_j^{k+1}$, thus $\epsilon_{i,1} \circ \lambda_\sigma = \lambda_{\tau^1_{\sigma,i}} \circ \epsilon_0$.

Let $k \in \{0, \dots, n\}$, we want to show that $(\epsilon_{i,0} \circ \lambda_\sigma)(k) = (\lambda_{\tau^0_{\sigma,i}} \circ \epsilon_n)(k)$. We have

$$(\epsilon_{i,0} \circ \lambda_\sigma)(k) = \epsilon_{i,0}(\alpha_1^k, \dots, \alpha_n^k) = (\gamma_1, \dots, \gamma_{n+1})$$

where

$$\gamma_j = \begin{cases} \alpha_j^k & \text{if } j < i \\ 0 & \text{if } i = j \\ \alpha_{j-1}^k & \text{if } j > i \end{cases}$$

and

$$(\lambda_{\tau^0_{\sigma,i}} \circ \epsilon_{n+1})(k) = \lambda_{\tau^0_{\sigma,i}}(k) = (\beta_1^k, \dots, \beta_{n+1}^k)$$

where $\beta_j^k = 1$ if and only if $j \in \{\tau^0_{\sigma,i}(1), \dots, \tau^0_{\sigma,i}(k)\}$. Then $\beta_j^k = 1$ if and only if there exists $l \in \{1, \dots, k\}$ such that $j = (\tau^0_{\sigma,i})(l)$.
If $j < i$, then $\beta_j^k = 1$ if and only if there exists $l \in \{1, \dots, k\}$ such that $j = \sigma(l)$.
If $j = i$, then $\beta_i^k = 0$ because it does not exist $l \in \{1, \dots, k\}$ such that $\tau^0_{\sigma,i}(l) = i$.
If $j > i$, then $\beta_j^k = 1$ if and only if there exists $l \in \{1, \dots, k\}$ such that $j = \sigma(l) + 1$.
Hence for all j, $\gamma_j = \beta_j^{k+1}$, thus $\epsilon_{i,0} \circ \lambda_\sigma = \lambda_{\tau^0_{\sigma,i}} \circ \epsilon_{n+1}$.

$\square$

Proof of the proposition: Let $n \in \mathbb{N}$, we want to show that the following diagram commutes

$$\begin{tikzcd} E^n(\Sigma_G(G), A) \arrow[r, "\Lambda^n"] \arrow[d, "d^n_\Sigma"'] & E^n(\Gamma_G(G), A) \arrow[d, "d^n_\Gamma"] \\ E^{n+1}(\Sigma_G(G), A) \arrow[r, "\Lambda^{n+1}"] & E^{n+1}(\Gamma_G(G), A) \end{tikzcd}$$

Let $f \in E^n(\Sigma_G(G), A)$ and $F \in \Sigma^n_G(G)$, we have

$$\begin{aligned} d^n_\Gamma(\Lambda^n(f))(F) &= \sum_{i=1}^n (-1)^{i+1}(d^{i,1}_\Gamma \Lambda^n(f) - d^{i,0}_\Gamma \Lambda^n(f))(F) \\ &= \sum_{i=1}^n (-1)^{i+1}(\Lambda^n(f) \circ \partial_{i,1} - \Lambda^n(f) \circ \partial_{i,0})(F) \\ &= \sum_{i=1}^n (-1)^{i+1}\big(\sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)}(f \circ \Lambda^n_\sigma \circ \partial_{i,1})(F) - \sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)}(f \circ \Lambda^n_\sigma \circ \partial_{i,0})(F)\big) \\ &= \sum_{i=1}^n (-1)^{i+1}\big(\sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)}(f \circ \Lambda^n_\sigma)(F \circ \epsilon_{i,1}) - \sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)}(f \circ \Lambda^n_\sigma)(F \circ \epsilon_{i,0})\big) \\ &= \sum_{i=1}^n (-1)^{i+1}\big(\sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)} f(F \circ \epsilon_{i,1} \circ \lambda^n_\sigma) - \sum_{\sigma \in \mathfrak{S}_n} (-1)^{sign(\sigma)} f(F \circ \epsilon_{i,0} \circ \lambda^n_\sigma)\big). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Lambda^{n+1}(d^n_\Sigma(f))(F) &= \sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}(d^n_\Sigma(f)\circ\Lambda^{n+1}_\sigma)(F)\\
&= \sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)} d^n_\Sigma(f)(F\circ\lambda^{n+1}_\sigma)\\
&= \sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\big(\sum_{i=0}^{n}(-1)^i d^i_\Sigma f(F\circ\lambda^{n+1}_\sigma)\big)\\
&= \sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\big(\sum_{i=0}^{n}(-1)^i (f\circ\partial_i)(F\circ\lambda^{n+1}_\sigma)\big)\\
&= \sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\big(\sum_{i=0}^{n}(-1)^i f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_i)\big)\\
&= \sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\big(f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_0)+\sum_{i=1}^{n-1}(-1)^i f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_i)\\
&\quad + (-1)^n f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_n)\big).
\end{aligned}$$

First, we show that

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\sum_{i=1}^{n-1}(-1)^i f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_i) = 0.$$

We have

$$\begin{aligned}
\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)} f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_i) &= \sum_{\sigma(k)<\sigma(k+1)} (-1)^{sign(\sigma)} f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_i)\\
&\quad+\sum_{\sigma(k)>\sigma(k+1)} (-1)^{sign(\sigma)} f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_i)\\
&= \sum_{\sigma(k)<\sigma(k+1)} (-1)^{sign(\sigma)} f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_i)\\
&\quad+\sum_{\sigma(k)<\sigma(k+1)} (-1)^{sign(\sigma)+1} f(F\circ\lambda^{n+1}_{\sigma\circ(k\,k+1)}\circ\epsilon_i).
\end{aligned}$$

We have shown in the preceding lemma that $\lambda^{n+1}_\sigma\circ\epsilon_k = \lambda^{n+1}_{\sigma\circ(k\,k+1)}\circ\epsilon_k$, for all $1\leq k\leq n-1$. Thus, we have

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\sum_{i=1}^{n-1}(-1)^i f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_i) = 0.$$

Hence, we have

$$\Lambda^{n+1}(d^n_\Sigma(f))(F) = \sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\big(f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_0)+(-1)^n f(F\circ\lambda^{n+1}_\sigma\circ\epsilon_n)\big).$$

Now we have to show that

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\big(f(F\circ\lambda_\sigma^{n+1}\circ\epsilon_0) = \sum_{i=1}^{n}(-1)^{i+1}\sum_{\sigma\in\mathfrak{S}_n}(-1)^{sign(\sigma)} f(F\circ\epsilon_{i,1}\circ\lambda_\sigma^n),$$

and

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}(-1)^n f(F\circ\lambda_\sigma^{n+1}\circ\epsilon_n) = \sum_{i=1}^{n}(-1)^{i}\sum_{\sigma\in\mathfrak{S}_n}(-1)^{sign(\sigma)} f(F\circ\epsilon_{i,0}\circ\lambda_\sigma^n).$$

We have shown in the preceding lemma that $\epsilon_{i,1}\circ\lambda_\sigma^n = \lambda^{n+1}_{\tau^1_{\sigma,i}}\circ\epsilon_0$. We remark that for all $\sigma\in\mathfrak{S}_{n+1}$, there exists a unique pair $(\beta,i)\in\mathfrak{S}_n\times\{1,\dots,n\}$ such that $\tau^1_{\beta,i}=\sigma$ and $sign(\beta)+i+1 = sign(\sigma)$. Hence we have

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_{n+1}} (-1)^{sign(\sigma)}\big(f(F\circ\lambda_\sigma^{n+1}\circ\epsilon_0) &= \sum_{\beta\in\mathfrak{S}_n,1\le i\le n}(-1)^{sign(\beta)+i+1}\big(f(F\circ\lambda^{n+1}_{\tau^1_{\beta,i}}\circ\epsilon_0)\\ &= \sum_{i=1}^{n}(-1)^{i+1}\sum_{\beta\in\mathfrak{S}_n}(-1)^{sign(\beta)} f(F\circ\epsilon_{i,1}\circ\lambda_\sigma^n).\end{aligned}$$

By the same argument, we show the second equality. Thus, $\{\Lambda^n\}_{n\in\mathbb{N}}$ is a morphism of cochain complexes.

□

The following proposition is one of the goals of this appendix. It establishes that the morphisms constructed above between $E^n(\Sigma_G(G),A)$ and $E^n(\Gamma_G(G),A)$, are the same as these defined in Proposition 2.3.24.

Proposition A.7.9. *For all $n\in\mathbb{N}$, $t^n_\Delta\circ\Lambda^n\circ((t^n_\square)^{-1}) = \Delta^n$.*

The proof is clear when we have shown the following lemma.

Lemma A.7.10. *Let $(g_0,\dots,g_n)\in G^{n+1}$ and $\sigma\in\mathfrak{S}_n$. We have $(t^n_\Delta\circ\Lambda^n_\sigma\circ(t^n_\square)^{-1})(g_0,\dots,g_n)$ is equal to $(h^\sigma_0,\dots,h^\sigma_n)$, where $h^\sigma_0 = g_0$ and $h^\sigma_k = g_{j_1}\triangleright\dots\triangleright g_{j_l}\triangleright g_{\sigma(k)}$ with $j_1<\dots<j_l<\sigma_k$ and $j_i\notin\{\sigma(1),\dots,\sigma(k)\}$.*

Proof : Let $(g_0,\dots,g_n)\in G^{n+1}$ and $F=(t^n_\square)^{-1}(g_0,\dots,g_n)$, the functor from $\square^+_n$ from $\mathcal{T}_G(G)$ corresponding to this $(n+1)$-tuple. By definition of the bijection t^n_Δ between $\Sigma^n_G(G)$ and G^{n+1}, we have $h^\sigma_0 = \Lambda^n_\sigma(F)(0) = F(\emptyset) = g_0$ and $h^\sigma_k = \Lambda^n_\sigma(F)((k-1)\to k)$. We are going to show the result by induction on k.

First, we have to initialise the induction. We have $h^\sigma_1 = \Lambda^n_\sigma(F)(0\to 1) = F(\emptyset)\to F(\lambda_\sigma(1)) = g_0 \to F(\alpha^\sigma_1,\dots,\alpha^\sigma_n)$ where there exists a unique α^σ_k equal to 1. If we use the subset description of the trunk $\square^+_n$, all the squares in the following diagram are preferred

$$\begin{tikzcd}
\{k\} \arrow[r, "1"] & \{1,k\} \arrow[r, "2"] & \{1,2,k\} \arrow[r] & \cdots \arrow[r] & \{1,\dots,k-2,k\} \arrow[r, "k-1"] & \{1,\dots,k\} \\
\emptyset \arrow[u, "k"] \arrow[r, "1"] & \{1\} \arrow[u, "k"] \arrow[r, "2"] & \{1,2\} \arrow[u, "k"] \arrow[r] & \cdots \arrow[r] & \{1,\dots,k-2\} \arrow[u, "k"] \arrow[r, "k-1"] & \{1,\dots,k-1\} \arrow[u, "k"]
\end{tikzcd}$$

Hence, we apply the trunk map F, we obtain

$$\begin{tikzcd}
F(\{k\}) \arrow[r, "g_1"] & F(\{1,k\}) \arrow[r, "g_2"] & F(\{1,2,k\}) \arrow[r] & \cdots \arrow[r] & F(\{1,\dots,k-2,k\}) \arrow[r, "g_{k-1}"] & F(\{1,\dots,k\}) \\
g_0 \arrow[u, "h_1^\sigma"] \arrow[r, "g_1"] & F(\{1\}) \arrow[u, "F(k)"] \arrow[r, "g_2"] & F(\{1,2\}) \arrow[u, "F(k)"] \arrow[r] & \cdots \arrow[r] & F(\{1,\dots,k-2\}) \arrow[u, "F(k)"] \arrow[r, "g_{k-1}"] & F(\{1,\dots,k-1\}) \arrow[u, "g_k"']
\end{tikzcd}$$

and by definition of the trunk $\mathcal{T}_G(G)$, we have

$$h_1^\sigma = g_1 \triangleright g_2 \triangleright \dots \triangleright g_{k-1} \triangleright g_k.$$

Secondly, suppose that the result is true for k, that is $h_k^\sigma = g_{j_1} \triangleright \dots \triangleright g_{j_l} \triangleright g_{\sigma(k)}$ with $j_1 < \cdots < j_l < \sigma_k$ and $j_i \notin \{\sigma(1), \dots, \sigma(k)\}$. Is it always true for $k+1$?
Suppose first that $\sigma(k) > \sigma(k+1)$. We denote V_k^σ the set of indexes $i \in \{1, \dots, n\}$ such that $\alpha_i^k \in \lambda_\sigma^n(k)$ is equal to 1. Then, using the subset description of $\square_n^+$, the following square is preferred in $\square_n^+$

$$\begin{tikzcd}
V_k^\sigma \arrow[r, "\sigma(k+1)"] & V_{k+1}^\sigma \\
V_{k-1}^\sigma \arrow[u, "\sigma(k)"] \arrow[r, "\sigma(k+1)"] & V_{k-1}^\sigma \cup \{\sigma(k+1)\} \arrow[u, "\sigma(k)"']
\end{tikzcd}$$

and the image of this preferred square by F is

$$\begin{tikzcd}
F(V_k^\sigma) \arrow[r, "h_{k+1}^\sigma"] & F(V_{k+1}^\sigma) \\
F(V_{k-1}^\sigma) \arrow[u, "h_k^\sigma"] \arrow[r, "F(\sigma(k+1))"] & F(V_{k-1}^\sigma \cup \{\sigma(k+1)\}) \arrow[u, "F(\sigma(k))"']
\end{tikzcd}$$

Because of F is a trunk map, this square is preferred in $\mathcal{T}_G(G)$, thus, by definition of the trunk $\mathcal{T}_G(G)$, we have $h_{k+1}^\sigma = F(\sigma(k+1)) : F(V_{k-1}^\sigma) \to F(V_{k-1}^\sigma \cup \{\sigma(k+1)\})$. We remark that $V_{k-1}^\sigma \overset{\sigma(k+1)}{\longrightarrow} V_{k-1}^\sigma \cup \{\sigma(k+1)\}$ is equal to $V_{k-1}^{\sigma\circ(k\,k+1)} \overset{\sigma\circ(k\,k+1)}{\longrightarrow} V_k^{\sigma\circ(k\,k+1)}$. Thus,

$$h_{k+1}^\sigma = h_k^{\sigma\circ(k\,k+1)}.$$

By induction hypothesis, we have $h_k^{\sigma\circ(k\,k+1)} = g_{j_1} \triangleright \dots \triangleright g_{j_l} \triangleright g_{\sigma(k+1)}$ with $j_1 < \cdots < j_l < \sigma(k+1)$ and $j_i \notin \{\sigma(1), \dots, \sigma(k+1)\}$. Hence, we have shown the result when $\sigma(k) > \sigma(k+1)$.
Suppose now that $\sigma(k) < \sigma(k+1)$. The following square is preferred in $\square_n^+$

$$\begin{tikzcd}
V_{k-1}^\sigma \cup \{\sigma(k+1)\} \arrow[r, "\sigma(k)"] & V_{k+1}^\sigma \\
V_{k-1}^\sigma \arrow[u, "\sigma(k+1)"] \arrow[r, "\sigma(k)"] & V_k^\sigma \arrow[u, "\sigma(k+1)"']
\end{tikzcd}$$

and the image of this square by F is the preferred square

$$\begin{tikzcd}
F(V_{k-1}^\sigma \cup \{\sigma(k+1)\}) \arrow[r, "F(\sigma(k))"] & F(V_{k+1}^\sigma) \\
F(V_{k-1}^\sigma) \arrow[u, "F(\sigma(k+1))"] \arrow[r, "h_k^\sigma"] & F(V_k^\sigma) \arrow[u, "h_{k+1}^\sigma"']
\end{tikzcd}$$

Hence, by the definition of the trunk $\mathcal{T}_G(G)$, we have $h^\sigma_{k+1} = (h^\sigma_k)^{-1} \triangleright F(\sigma(k+1))$. Like in the precedent case, we have $F(\sigma(k+1)) = h_k^{\sigma\circ(k\,k+1)}$, thus

$$h^\sigma_{k+1} = (h^\sigma_k)^{-1} \triangleright h_k^{\sigma\circ(k\,k+1)}.$$

By induction hypothesis, we have $h_k^{\sigma\circ(k\,k+1)} = g_{j_1} \triangleright \ldots \triangleright g_{j_l} \triangleright g_{\sigma(k+1)}$ with $j_1 < \cdots < j_l < \sigma(k+1)$ and $j_i \notin \{\sigma(1), \ldots, \sigma(k-1), \sigma(k+1)\}$. That is,

$$h_k^{\sigma\circ(k\,k+1)} = g_{j_1} \triangleright \ldots \triangleright g_{j_l} \triangleright g_{\sigma(k)} \triangleright g_{\sigma(k+1)},$$

and so

$$\begin{aligned}
h^\sigma_k &= (h^\sigma_k)^{-1} \circ h_k^{\sigma\circ(k\,k+1)} \\
&= (g_{j_1} \triangleright \ldots \triangleright g_{j_l} \triangleright g_{\sigma(k)})^{-1} \triangleright (g_{j_1} \triangleright \ldots \triangleright g_{j_l} \triangleright g_{\sigma(k)} \triangleright g_{\sigma(k+1)}) \\
&= (g_{j_1} \triangleright \ldots \triangleright g_{j_l} \triangleright (g_{\sigma(k)})^{-1}) \triangleright (g_{j_1} \triangleright \ldots \triangleright g_{j_l} \triangleright g_{\sigma(k)} \triangleright g_{\sigma(k+1)}) \\
&= g_{j_1} \triangleright \ldots \triangleright g_{j_l} \triangleright ((g_{\sigma(k)})^{-1} \triangleright (g_{\sigma(k)} \triangleright g_{\sigma(k+1)})) \\
&= g_{j_1} \triangleright \ldots \triangleright g_{j_l} \triangleright g_{\sigma(k+1)}.
\end{aligned}$$

with $j_1 < \cdots < j_l < \sigma(k+1)$ and $j_i \notin \{\sigma(1), \ldots, \sigma(k-1), \sigma(k), \sigma(k+1)\}$. Hence we have shown the lemma.

$\square$

Appendix B

Synthèse en français

Introduction

Le résultat principal de cette thèse est une solution locale du *problème des coquecigrues*. Par problème des coquecigrues, nous parlons du problème d'intégration des algèbres de Leibniz. Cette question a été posé par J.-L. Loday dans [Lod93] et consiste à généraliser le troisième théorème de Lie aux algèbres de Leibniz. Ce théorème établit que pour toutes algèbres de Lie $\mathfrak{g}$, il existe un groupe de Lie G tel que son espace tangent en 1 soit muni d'une structure d'algèbre de Lie isomorphe à $\mathfrak{g}$. Les algèbres de Leibniz sont des généralisations des algèbres de Lie, ce sont leurs analogues non-commutatifs. Précisemment, une *algèbre de Leibniz (à gauche)* (sur $\mathbb{R}$) est un $\mathbb{R}$-espace vectoriel $\mathfrak{g}$ muni d'une forme bilinéaire $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ appelée le crochet et satisfaisant l'*identité de Leibniz (à gauche)* pour tout x, y et z appartenant à $\mathfrak{g}$

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$

Ainsi, une question naturelle est de savoir si pour toutes algèbres de Leibniz, il existe une variété lisse munie d'une structure algébrique généralisant la structure de groupe, et telle que l'espace tangent en un point donné, appelé élément neutre et noté 1, peut être muni d'une structure d'algèbre de Leibniz isomorphe à l'algèbre de Leibniz donnée. Comme nous voulons que cette intégration soit une généralisation du cas des algèbres de Lie, nous devons aussi demander que, quand l'algèbre de Leibniz est une algèbre de Lie, la variété intégrante soit un groupe de Lie.

Le résultat principal concernant ce problème a été donné par M.K. Kinyon dans [Kin07]. Dans cette article, il résout le cas des algèbres de Leibniz *scindées*, c'est à dire les algèbres de Leibniz qui sont *produits demisemidirectes* d'une algèbre de Lie et d'une représentation sur cette algèbre de Lie. C'est à dire, les algèbres de Leibniz qui sont isomorphe à $\mathfrak{g} \oplus \mathfrak{a}$ en tant qu'espace vectoriel, et où le crochet est donné par $[(x,a),(y,b)] = ([x,y], x.a)$. Dans ce cas, il montre que la structure algébrique répondant au problème est la structure de *digroupe*. Un digroupe est un ensemble muni de deux produits binaires $\vdash$ et $\dashv$, un élément neutre 1 et des relations de compatibilités. Plus précisemment, il montre qu'une structure de digroupe induit une structure de *rack pointé* (pointé en 1), et c'est cette structure algébrique qui munit l'espace tangent en 1 d'une structure d'algèbre de Leibniz. Bien sûr, toutes les algèbres de Leibniz ne sont pas isomorphe à un produit demisemidirecte, donc nous devons trouver une structure plus générale pour résoudre ce problème. Nous pouvons penser que la bonne structure est celle de rack pointé, mais M.K. Kinyon a montré dans [Kin07] que la seconde condition (une algèbre de Lie s'intègre en un groupe de Lie) n'est pas toujours remplie. Donc nous devons spécifier la structure dans la catégorie des racks pointés.

Dans cette thèse nous ne donnons pas une réponse complète au problème des coquecigrues, dans le sens où nous construisons seulement une structure algébrique *locale* et non pas globale. En effet, pour définir une structure algébrique sur l'espace tangent en un point donné, nous avons juste besoin d'une structure algébrique au voisinage de ce point. Nous montrerons dans la section B.3 que la solution locale à ce problème est donnée par les racks de Lie augmentés locaux.

Notre approche du problème est similaire à celle donnée par E. Cartan dans [Car30]. L'idée principale est que nous connaissons le troisième théorème de Lie pour certaines classes d'algèbres de Lie. Par exemple, une algèbre de Lie abélienne s'intègre en elle même, et en utilisant le premier théorème de Lie, une sous algèbre de Lie de l'algèbre de Lie $End(V)$ s'intègre en un sous groupe de Lie de $Gl(V)$. Soit $\mathfrak{g}$ une algèbre de Lie, $Z(\mathfrak{g})$ son centre et $\mathfrak{g}_0$ le quotient de $\mathfrak{g}$ par $Z(\mathfrak{g})$. L'algèbre de Lie $Z(\mathfrak{g})$ est abélienne et $\mathfrak{g}_0$ est une sous algèbre de Lie de $End(\mathfrak{g})$, donc il existe des groupes de Lie, respectivement $Z(\mathfrak{g})$ et G_0, qui intègrent ces algèbres de Lie. Comme espace vectoriel, $\mathfrak{g}$ est isomorphe à la somme directe $\mathfrak{g}_0 \oplus Z(\mathfrak{g})$, donc l'espace tangent en $(1,0)$ de la variété $G_0 \times Z(\mathfrak{g})$ est isomorphe à $\mathfrak{g}$. Comme algèbre de Lie, $\mathfrak{g}$ est isomorphe à l'extension centrale $\mathfrak{g}_0 \oplus_\omega Z(\mathfrak{g})$ où ω est un 2-cocycle de Lie sur $\mathfrak{g}_0$ à coefficients dans $Z(\mathfrak{g})$. C'est à dire, le crochet sur $\mathfrak{g}_0 \oplus_\omega Z(\mathfrak{g})$ est défini par

$$[(x,a),(y,b)] = \big([x,y], \omega(x,y)\big), \tag{B.1}$$

où ω est une forme bilinéaire antisymétrique $\mathfrak{g}_0$ à valeurs dans $Z(\mathfrak{g})$ qui satisfait l'identité de cocycle d'algèbre de Lie

$$\omega([x,y],z) - \omega(x,[y,z]) + \omega(y,[x,z]) = 0.$$

Ainsi nous devons trouver une structure de groupe sur $G_0 \times Z(\mathfrak{g})$ qui donne cette structure d'algèbre de Lie sur l'espace tangent en $(1,0)$. Il est clair que le crochet (B.1) est complètement déterminé par le crochet sur $\mathfrak{g}_0$ et le cocycle ω. Ainsi, la seule chose que nous avons à comprendre est ω. L'algèbre de Lie $\mathfrak{g}$ est une extension centrale de $\mathfrak{g}_0$ par $Z(\mathfrak{g})$, donc nous pouvons espérer que le groupe de Lie intégrant $\mathfrak{g}$ soit une extension central de G_0 par $Z(\mathfrak{g})$. Pour suivre cette idée, nous devons trouver un 2-cocycle de groupe sur G_0 à coefficients dans $Z(\mathfrak{g})$. Dans ce cas, la structure de groupe sur $G_0 \times Z(\mathfrak{g})$ est donnée par

$$(g,a).(h,b) = \big(gh, a+b+f(g,h)\big), \tag{B.2}$$

où f est une application de $G \times G$ vers $Z(\mathfrak{g})$ s'annulant en $(1,g)$ et $(g,1)$ et satisfaisant l'identité de cocycle de groupe

$$f(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0.$$

Avec un tel cocycle, la conjugaison dans le groupe est donnée par

$$(g,a).(h,b).(g,a)^{-1} = \big(ghg^{-1}, a + f(g,h) - f(ghg^{-1},g)\big), \tag{B.3}$$

et en imposant une condition de différentiabilité sur f dans un voisinage de 1, alors nous pouvons dériver cette formule deux fois, et obtenir un crochet sur $\mathfrak{g}_0 \oplus Z(\mathfrak{g})$ défini par

$$[(x,a),(y,b)] = \big([x,y], D^2 f(x,y)\big),$$

où $D^2 f(x,y) = d^2 f(1,1)((x,0),(0,y)) - d^2 f(1,1)((y,0),(0,x))$. Donc, si $D^2 f(x,y)$ est égal à $\omega(x,y)$, alors nous retrouvons le crochet (B.1). Ainsi, si nous associons à ω un cocycle de groupe f satisfaisant des hypothèses de différentiabilité et tel que $D^2 f = \omega$, alors notre problème d'intégration est résolu. Ceci peut être fait en deux étapes. La première consiste à trouver un cocycle de groupe de Lie local défini au voisinage de 1. Précisemment, nous voulons une application f défini sur un sous ensemble de $G_0 \times G_0$ contenant $(1, 1)$ à valeurs dans $Z(\mathfrak{g})$ qui satisfait l'identité de cocycle de groupe local (cf. [Est54] pour la définition d'un groupe local). Nous pouvons construire explicitement un tel cocycle de groupe local. Cette construction est la suivante (cf. Lemma 5.2 dans [Nee04]).

Soit V un voisinage ouvert convexe de 0 dans $\mathfrak{g}$, et $\phi : V \to G_0$ une carte de G_0 avec $\phi(0) = 1$ et $d\phi(0) = id_{\mathfrak{g}_0}$. Pour tout $(g, h) \in \phi(V) \times \phi(V)$ tel que $gh \in \phi(V)$, définissons $f(g, h) \in Z(\mathfrak{g})$ par la formule

$$f(g, h) = \int_{\gamma_{g,h}} \omega^{inv}$$

où $\omega^{inv} \in \Omega^2(G_0, Z(\mathfrak{g}))$ est la forme différentielle invariante sur G_0 associée à ω et $\gamma_{g,h}$ est la 2-chaine singulière lisse définie par

$$\gamma_{g,h}(t, s) = \phi(t(\phi^{-1}(g\phi(s\phi^{-1}(h)))) + s(g\phi((1-t)\phi^{-1}(h)))).$$

Cette formule définit une fonction lisse vérifiant $D^2 f(x, y) = \omega(x, y)$. Nous devons maintenant vérifier que f satisfait l'identité de cocycle de groupe local. Soit $(g, h, k) \in \phi(V)^3$ tel que gh, hk and ghk sont dans $\phi(V)$. Nous avons

$$\begin{aligned} f(h, k) - f(gh, k) + f(g, hk) - f(g, h) &= \int_{\gamma_{h,k}} \omega^{inv} - \int_{\gamma_{gh,k}} \omega^{inv} + \int_{\gamma_{g,hk}} \omega^{inv} - \int_{\gamma_{g,h}} \omega^{inv} \\ &= \int_{\partial\gamma_{g,h,k}} \omega^{inv} \end{aligned}$$

où $\gamma_{g,h,k}$ est une 3-chaine singulière lisse à valeurs dans $\phi(V)$ vérifiant $\partial\gamma_{g,h,k} = g\gamma_{h,k} - \gamma_{gh,k} + \gamma_{g,hk} + \gamma_{g,h}$ (une telle chaine existe car $\phi(V)$ est homéomorphe à un sous ensemble ouvert convexe V de $\mathfrak{g}_0$). Donc

$$\begin{aligned} f(h, k) - f(gh, k) + f(g, hk) - f(g, h) &= \int_{\partial\gamma_{g,h,k}} \omega^{inv} \\ &= \int_{\gamma_{g,h,k}} d_{dR}\omega^{inv} \\ &= 0 \end{aligned}$$

car ω^{inv} est une 2-forme fermée. Ainsi, nous avons associé à ω un 2-cocycle de groupe local, lisse dans un voisinage de 1, et tel que $D^2 f(x, y) = \omega(x, y)$. Donc nous définissons une structure de groupe de Lie local sur $G_0 \times Z(\mathfrak{g})$ en posant

$$(g, a).(h, b) = \big(gh, a + g.b + f(g, h)\big),$$

et l'espace tangent en $(1, 0)$ de ce groupe de Lie local est isomorphe à $\mathfrak{g}$. Si nous voulons une structure globale, nous devons étendre ce cocycle local au groupe G_0 en entier. Tout d'abord P.A. Smith ([Smi50, Smi51]), puis W.T. Van Est ([Est62]) ont montré que c'est précisemment cette globalisation qui rencontre une obstruction venant à la fois du $\pi_2(G_0)$ et $\pi_1(G_0)$.

Pour intégrer les algèbres de Leibniz en rack de Lie pointé, nous suivons une approche similaire. Dans ce contexte, nous utilisons le fait que nous savons intégrer les algèbres de Lie de dimension finie. Comme dans le cas des algèbres de Lie, nous associons à toutes algèbres de Leibniz une extension abélienne d'une algèbre de Lie $\mathfrak{g}_0$ par une représentation antisymétrique $Z_L(\mathfrak{g})$. Comme nous avons le théorème pour les algèbres de Lie, nous pouvons intégrer $\mathfrak{g}_0$ et

$Z_L(\mathfrak{g})$ en les groupes de Lie G_0 et $Z_L(\mathfrak{g})$. En utilisant le second théorème de Lie, on montre que $Z_L(\mathfrak{g})$ est un G_0-module. Ainsi, la difficulté principale devient l'intégration du cocycle de Leibniz en cocycle de rack de Lie local. Dans la section B.3 nous expliquons comment résoudre ce problème. Nous effectuons une construction similaire au cas des algèbres de Lie, mais dans ce contexte, il y a plusieurs difficultés qui apparaissent. Une d'entre elles est que notre cocycle de Leibniz n'est pas antisymétrique, donc nous ne pouvons considérer la forme différentielle équivariante associée et l'intégrer. Pour résoudre ce problème, nous utiliserons la Proposition B.1.1, qui en particulier établit un isomorphisme entre le second groupe de cohomologie d'une algèbre de Leibniz $\mathfrak{g}$ à coefficients dans une représentation antisymétrique $\mathfrak{a}^a$ et le premier groupe de cohomologie de $\mathfrak{g}$ à coefficients dans la représentation symétrique $Hom(\mathfrak{g},\mathfrak{a})^s$. De cette façon, nous obtenons une 1-forme que nous pouvons maintenant intégrer. Une autre difficulté est de spécifier sur quelle domaine cette 1-forme doit être intégrer. Dans le cas des algèbres de Lie, nous intégrons sur un 2-simplexe et l'identité de cocycle est vérifiée en intégrant sur un 3-simplexe, tandis que dans notre contexte nous remplacerons le 2-simplexe par un 2-cube et le 3-simplexe par un 3-cube.

Section 1: Algèbres de Leibniz

Ce chapitre entier, excepté la derniere proposition, est basé sur [Lod93, LP93, Lod98]. Nous donnons tout d'abord les définitions basiques dont nous avons besoin sur les algèbres de Leibniz. Contrairement à J.-L. Loday et T. Pirashvili, qui travaillent avec des algèbres de Leibniz à droite, nous étudions les algèbres de Leibniz à gauche. Ainsi, nous devons traduire toutes les définitions dont nous avons besoin dans notre contexte. Comme nous l'avons vu ci-dessus, nous traduisons notre problème d'intégration en un problème cohomologique, donc nous avons besoin d'une théorie de cohomologie pour les algèbres de Leibniz et, a fortiori, une notion de représentation. Nous prenons la définition d'une représentation d'une algèbre de Leibniz donné par J.-L. Loday et T. Pirashvili dans [LP93]. En particulier, ils montrent dans cette article l'équivalence entre la catégorie des représentations d'une algèbre de Leibniz et la catégorie des modules sur une algèbre associative notée $UL(\mathfrak{g})$. Toujours en suivant [LP93], nous définissons le complexe de cochaînes de Leibniz d'une algèbre de Leibniz à coefficients dans une représentation et nous décrivons HL^0, HL^1 et HL^2. En particulier, nous montrons que HL^0 correspond aux *invariants à droite*, HL^1 correspond aux *dérivations* modulo les *dérivations intérieurs* et HL^2 correspond aux *extensions abéliennes*. Ensuite, nous donnons des exemples d'extensions abéliennes associées à une algèbre de Leibniz (*extension caractéristique*, *extension par le centre à gauche* et *extension par le centre*). Nous finissons par une proposition fondamentale (Proposition B.1.1), établissant un isomorphisme du complexe de cochaînes de $\{CL^*(\mathfrak{g},\mathfrak{a}^a),d_L^*\}_{n\in\mathbb{N}}$ vers $\{CL^{n-1}(\mathfrak{g},Hom(\mathfrak{g},\mathfrak{a})^s)\}_{n\in\mathbb{N}}$. Ce qui est important dans ce résultat est le passage d'une représentation antisymétrique à une représentation symétrique. Ceci nous permettra d'associer un 2-cocycle de rack de Lie local à un 2-cocycle de Leibniz.

Section 2: Racks de Lie

La notion de racks vient de la topologie, en particulier de la théorie des invariants des noeuds et des entrelacs (cf. par exemple [FR]). C'est M.K. Kinyon dans [Kin07] qui a été le premier à relier les racks et les algèbres de Leibniz. L'idée de relier ces deux structures vient de la théorie des groupes et algèbres de Lie, en particulier de la construction du crochet en utilisant la conjugaison. En effet, une façon de définir un crochet sur l'espace tangent en 1 d'un groupe de Lie est de dériver deux fois le morphisme de conjugaison. Soit G un groupe de Lie, la conjugaison est le morphisme de groupe $c: G \to Aut(G)$ défini par $c_g(h) = ghg^{-1}$. Si l'on dérive cette expression par rapport à la variable h en 1, nous obtenons un morphisme de groupes de Lie $Ad : G \to Aut(\mathfrak{g})$. Nous pouvons encore dériver ce morphisme en 1 pour obtenir une application linéaire $ad : \mathfrak{g} \to End(\mathfrak{g})$. Alors, nous sommes en mesure de définir un crochet $[-,-]$ sur $\mathfrak{g}$ en posant $[x,y] = ad(x)(y)$. Nous pouvons montrer que ce crochet satisfait l'identité de Leibniz à gauche, et que cette identité est induite par l'égalité $c_g(c_h(k)) = c_{c_g(h)}(c_g(k))$. Donc, si nous notons $c_g(h)$ par $g \rhd h$, les seules propriétés que nous utilisons pour définir un crochet de Lie sur $\mathfrak{g}$ sont

1. $g \rhd : G \to G$ est une bijection pour tout $g \in G$.
2. $g \rhd (h \rhd k) = (g \rhd h) \rhd (g \rhd k)$ pour tout $g, h, k \in G$
3. $g \rhd 1 = 1$ et $1 \rhd g = g$ pour tout $g \in G$.

Ainsi, nous appelons *rack (à gauche)*, un ensemble muni d'une opération binaire $\rhd$ satisfaisant la première et seconde condition. Un rack est dit *pointé* si il existe un élément 1 qui satisfait la troisième condition. Nous commencons ce chapitre en donnant des définitions et exemples, pour ceci nous suivons [FR]. Ils travaillent avec des racks à droite, ainsi comme dans le cas des algèbres de Leibniz, nous traduisons les définitions dans le contexte des racks à gauche. En particulier, nous donnons le plus important exemple de rack appelé *rack augmenté (pointé)* (cf. section B.2.1). Cet exemple présente des similarités avec les modules croisés de groupe, et dans ce cas la structure de rack est induite par une action de groupe.

Comme dans le cas des groupes, nous voulons construire le rack pointé associé à une algèbre de Leibniz en utilisant une extension abélienne. Ainsi, nous avons besoin d'une théorie de cohomologie où le second groupe de cohomologie correspond aux classes d'extensions du rack par un module. Dans [Jac07], N. Jackson donne des définitions très générales de module et de cohomologie qui généralisent celles données tout d'abord par P. Etingof et M. Graña dans [EG03], puis par N. Andruskiewitsch et M. Graña dans [AG03]. Nous n'avons pas besoin d'un tel degré de généralité, et nous prenons comme définitions celles données dans [AG03]. Avec ces définitions et en traduisant dans le contexte des racks à gauche une preuve donnée par N. Jackson dans [Jac07], on peut alors montrer que le second groupe de cohomologie classifie les classes d'équivalences d'extensions abéliennes. La version pointé de ce théorème s'en déduit alors facilement.

Nous finissons ce chapitre en donnant les définitions de cohomologie de rack local et cohomologie de rack de Lie (local).

Section 3: Racks de Lie et algèbres de Leibniz

Ce chapitre est le coeur de notre thèse. Il donne une solution locale au problème des coquecigrues. À notre connaissance, tous les résultats de ce chapitre sont nouveau excepté la Proposition B.3.1 due à M.K. Kinyon ([Kin07]). Tout d'abord, nous rappelons le lien entre racks de Lie (local) et algèbres de Leibniz expliqué par M.K. Kinyon dans [Kin07] (Proposition B.3.1). Ensuite, nous étudions le passage des $As_p(X)$-modules lisses aux représentations de Leibniz (Proposition B.3.6) et le passage de la cohomologie de rack de Lie (local) à la cohomologie de Leibniz. Nous définissons un morphisme de la cohomologie de la cohomologie de rack de Lie (local) d'un rack X à coefficients dans un $As_p(X)$-module A^s (resp. A^a) vers vers la cohomologie de Leibniz de l'algèbre de Leibniz associée à X à coefficients dans $\mathfrak{a}^s = T_0A$ (resp. $\mathfrak{a}^a$) (Proposition B.3.7). La fin de ce chapitre (sections B.3.4 et B.3.5) est sur l'intégration des algèbres de Leibniz en rack de Lie local. Nous utilisons la même approche que celle d'E. Cartan dans le cas des groupes . C'est à dire, pour toutes algèbres de Leibniz nous considérons l'extension abélienne par le centre à gauche et nous l'intégrons. Cette extension est caractérisée par un 2-cocycle, et nous construisons

(Proposition B.3.19) un 2-cocycle de rack de Lie local l'intégrant par une construcition explicite similaire à celle expliquée dans le cas des groupes de Lie au début de cette introduction. Cette construction est résumée dans notre principale théorème (Théorème B.3.25). Nous remarquons que le 2-cocycle construit a plus de structure (Proposition B.3.23). C'est à dire, l'identité de cocycle de rack est induite par une autre. Cette identité nous permet de munir notre rack de Lie local construit, d'une structure de rack de Lie augmenté local (Proposition B.3.28). Nous finissons ce chapitre avec des examples d'intégration d'algèbres de Leibniz non scindées en dimensions 4 et 5.

B.1 Algèbres de Leibniz

B.1.1 Définitions

Comme il est écrit dans l'introduction, nous travaillons avec des algèbres de Leibniz à gauche à la place d'algèbres de Leibniz à droite. La raison principale vient du fait que M.K. Kinyon travaille dans ce contexte dans son article [Kin07], et cet article est notre point de départ pour le problème d'intégration des algèbres de Leibniz. Ainsi, nous avons choisi de travailler dans ce contexte.
Une *algèbre de Leibniz (à gauche) (sur $\mathbb{R}$)* est un espace vectoriel $\mathfrak{g}$ (sur $\mathbb{R}$) muni d'un crochet $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, qui satisfait l'*identité de Leibniz à gauche*:

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

Remarquons qu'une façon équivalente de définir les algèbres de Leibniz à gauche est de dire que, pour tout $x \in \mathfrak{g}$, $[x,-]$ est une dérivation pour le crochet $[-,-]$. Le premier exemple d'algèbre de Leibniz est une algèbre de Lie. En effet, si le crochet est antisymétrique, alors l'identité de Leibniz est équivalente à l'identité de Jacobi. Ainsi, nous avons un foncteur $inc : Lie \to Leib$. Ce foncteur a un adjoint à gauche $(-)_{Lie} : Leib \to Lie$ qui est défini sur les objets par $\mathfrak{g}_{Lie} = \mathfrak{g}/ < [x,x] >$, où $< [x,x] >$ est l'idéal bilatère de $\mathfrak{g}$ engendré par l'ensemble $\{[x,x] \in \mathfrak{g} \,|\, x \in \mathfrak{g}\}$. On peut remarquer qu'il y a d'autres façons de construire une algèbre de Lie à partir d'une algèbre de Leibniz. Une possibilité est de quotienter $\mathfrak{g}$ par le *centre à gauche* $Z_L(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [x,-] = 0\}$, mais cette construction n'est plus fonctoriel.

B.1.2 Représentations de Leibniz

Pour définir une théorie de cohomologie pour les algèbres de Leibniz, nous avons besoin d'une notion de représentation d'une telle structure algébrique. Comme nous travaillons avec des algèbres de Leibniz à gauche, nous devons traduire la définition donnée par J.-L. Loday and T. Pirashvili dans leur article [LP93]. Dans notre contexte, une *représentation* d'une algèbre de Leibniz $\mathfrak{g}$, devient un espace vectoriel M muni de deux applications linéaires $[-,-]_L : \mathfrak{g} \otimes M \to M$ and $[-,-]_R : M \otimes \mathfrak{g} \to \mathfrak{g}$, satisfaisant les trois axiomes suivants

$$[x,[y,m]_L]_L = [[x,y],m]_L + [y,[x,m]_L]_L \tag{LLM}$$
$$[x,[m,y]_R]_L = [[x,m]_L,y]_R + [m,[x,y]]_R \tag{LML}$$
$$[m,[x,y]]_R = [[m,x]_R,y]_R + [x,[m,y]_R]_L \tag{MLL}$$

Rappelons que pour une algèbre de Lie $\mathfrak{g}$, une représentation de $\mathfrak{g}$ est un espace vectoriel M muni d'une application linéaire $[-,-] : \mathfrak{g} \otimes M \to M$ satisfaisant $[[x,y],m] = [x,[y,m]] - [y,[x,m]]$. Une algèbre de Lie est une algèbre de Leibniz, ainsi nous voulons qu'une représentation de Lie

M d'une algèbre de Lie $\mathfrak{g}$, soit une représentation de Leibniz sur $\mathfrak{g}$. Nous avons deux choix canoniques pour définir une structure de représentation de Leibniz sur M. Une possibilité est de poser $[-,-]_L = [-,-]$ et $[-,-]_R = -[-,-]$, et une seconde est de poser $[-,-]_L = [-,-]$ and $[-,-]_R = 0$. Ces représentations de Leibniz sont des exemples particuliers de représentations de Leibniz. La première est un exemple de représentation *symetrique*, et la deuxième est un exemple de représentation *antisymetrique*. Une représentation symétrique est une représentation de Leibniz où $[-,-]_L = -[-,-]_R$ et une représentation antisymétrique est une représentation de Leibniz où $[-,-]_R = 0$. Une représentation de Leibniz qui est symétrique et antisymétrique est dite *triviale*.

B.1.3 Cohomologie des algèbres de Leibniz

Maintenant, nous sommes prêts pour définir une théorie de cohomologie pour les algèbres de Leibniz. L'existence d'une théorie de cohomologie (et d'homologie) pour ces algèbres est la principale motivation de leur étude, car restreinte aux algèbres de Lie, cette théorie nous donne de nouveaux invariants (cf. [Lod93]). Pour $\mathfrak{g}$ une algèbre de Leibniz et M une représentation de $\mathfrak{g}$, nous définissons un complexe de cochaîne $\{CL^n(\mathfrak{g}, M), d_L^n\}_{n\in\mathbb{N}}$ en posant

$$CL^n(\mathfrak{g}, M) = Hom(\mathfrak{g}^{\otimes n}, M)$$

et

$$\begin{aligned} d_L^n\omega(x_0,\dots,x_n) &= \sum_{i=0}^{n-1}(-1)^i[x_i, \omega(x_0,\dots,\hat{x_i},\dots,x_n)]_L + (-1)^{n-1}[\omega(x_0,\dots,x_{n-1}),x_n]_R \\ &\quad + \sum_{0\leq i<j\leq n}(-1)^{i+1}\omega(x_0,\dots,x_{j-1},[x_i,x_j],x_{j+1},\dots,x_n) \end{aligned}$$

Pour prouver que $d_L^{n+1}\circ d_L^n = 0$, nous utilisons les *formules de Cartan*. Ces formules sont décrites dans [LP93] dans le contexte des algèbres de Leibniz à droite, mais nous pouvons facilement les adapter dans notre contexte.

Comme dans la plupart des structures algébriques, le second groupe de cohomologie d'une algèbre de Leibniz $\mathfrak{g}$ à coefficient dans une représentation M est en bijection avec l'ensemble des classes d'équivalences d'extensions abéliennes de $\mathfrak{g}$ par M (cf. [LP93]). *Une extension abélienne* d'une algèbre de Leibniz $\mathfrak{g}$ par M est une algèbre de Leibniz $\hat{\mathfrak{g}}$, telle que $M \overset{i}{\hookrightarrow} \hat{\mathfrak{g}} \overset{p}{\twoheadrightarrow} \mathfrak{g}$ est une suite exacte courte d'algèbres de Leibniz (où M est considérée comme une algèbre de Leibniz abélienne), et la structure de représentation de M est compatible avec la structure de représentation induite par cette suite exacte courte. C'est à dire, $[m,x]_R = i^{-1}[i(m), s(x)]$ and $[x,m]_L = i^{-1}[s(x), i(m)]$ où s est une section de p et le crochet est celui de $\hat{\mathfrak{g}}$ (bien sûr, nous devons justifier que cette structure de représentation sur $\mathfrak{g}$ de M induite par la suite exacte courte ne dépend pas de s, mais nous le déduisons facilement du fait que la différence de daux sections de p est dans $i(M)$).

Il y a des extensions abéliennes canoniques associées à une algèbre de Leibniz. Celle que nous utiliserons pour intégrer les algèbres de Leibniz est l'*extension abélienne par le centre à gauche*

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$$

où $\mathfrak{g}_0 := \mathfrak{g}/Z_L(\mathfrak{g})$. C'est une extension d'une algèbre de Lie par une représentation antisymétrique. En un sens, une représentation symétrique est plus proche d'une représentation de Lie qu'une représentation antisymétrique. Ainsi, il est pratique de pouvoir passer d'une représentation symétrique à une représentation antisymétrique. Soit $\mathfrak{g}$ une algèbre de Lie et M

une représentation de Lie de $\mathfrak{g}$, alors nous définissons une structure de représentation de Lie sur $Hom(\mathfrak{g}, M)$ en posant

$$(x.\alpha)(y) := x.(\alpha(y)) - \alpha([x, y])$$

pour tout $x, y \in \mathfrak{g}$ et $\alpha \in Hom(\mathfrak{g}, M)$. La proposition suivante établit un isomorphisme de $HL^n(\mathfrak{g}, M^a)$ vers $HL^{n-1}(\mathfrak{g}, Hom(\mathfrak{g}, M)^s)$.

Proposition B.1.1. *Soient $\mathfrak{g}$ une algèbre de Lie et M une représentation de Lie de $\mathfrak{g}$. Nous avons un isomorphisme de complexes de cochaines*

$$CL^n(\mathfrak{g}, M^a) \xrightarrow{\tau^n} CL^{n-1}(\mathfrak{g}, Hom(\mathfrak{g}, M)^s)$$

donné par $\omega \mapsto \tau^n(\omega)$ où

$$\tau^n(\omega)(x_1, \ldots, x_{n-1})(x_n) = \omega(x_1, \ldots, x_n)$$

Preuve : Ce morphisme est clairement un isomorphisme $\forall n \geq 0$. De plus, nous avons

$$\begin{aligned}
d_L\tau^n(\omega)(x_0, \ldots, x_{n-1})(x_n) &= \sum_{i=0}^{n-2}(-1)^i[x_i, \tau^n(\omega)(x_0, \ldots, \hat{x_i}, \ldots, x_{n-1})](x_n) \\
&\quad +(-1)^{n-1}[x_{n-1}, \tau^n(\omega)(x_0, \ldots, x_{n-2})](x_n) \\
&\quad + \sum_{0\leq i<j\leq n-1} (-1)^{i+1}\tau^n(\omega)(x_0, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_{n-1})(x_n) \\
&= \sum_{i=0}^{n-1}(-1)^i([x_i, \omega(x_0, \ldots, \hat{x_i}, \ldots, x_{n-1}, x_n)] - \omega(x_0, \ldots, \hat{x_i}, \ldots, x_{n-1}, [x_i, x_n])) \\
&\quad + \sum_{0\leq i<j\leq n-1} (-1)^{i+1}\omega(x_0, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_{n-1}, x_n) \\
&= \sum_{i=0}^{n-1}(-1)^i[x_i, \omega(x_0, \ldots, \hat{x_i}, \ldots, x_{n-1}, x_n)] \\
&\quad + \sum_{0\leq i<j\leq n} (-1)^{i+1}\omega(x_0, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_{n-1}, x_n) \\
&= d_L\omega(x_0, \ldots, x_{n-1}, x_n) \\
&= \tau^{n+1}(d_L\omega)(x_0, \ldots, x_{n-1})(x_n)
\end{aligned}$$

Ainsi $\{\tau^n\}_{n\geq 0}$ est un morphisme de complexes de cochaines.

□

Remark B.1.2. Cette proposition est une généralisation d'une remarque faite par T. Pirashvili dans son article [Pir94] (section 2 - Proposition 2.1). Cette proposition est aussi un cas particulier d'un résultat présent dans le mémoire de Master de B. Jubin (Corollaire 2.21 dans [Jub06]).

B.2 Racks de Lie

B.2.1 Définitions et exemples

Comme dans le cas des algèbres de Leibniz, on peut définir les racks à gauche et les racks à droite. Étant donné que nous avons choisi de travailler avec des algèbres de Leibniz à gauche, nous prenons la définition de racks à gauche. Un *rack (à gauche)* est un ensemble X muni d'un produit $\rhd : X \times X \to X$, qui satisfait l'*identité de rack à gauche*, c'est à dire pour tout $x, y, z \in X$

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z),$$

et tel que $c_x := x \rhd \_ : X \to X$ est une bijection pour tout $x \in X$. Un rack est dit *pointé* si il existe un élément $1 \in X$, appelé l'élément neutre, qui satisfait $1 \rhd x = x$ et $x \rhd 1 = 1$ pour tout $x \in X$. Un *morphisme de rack* est une application $f : X \to Y$ satisfaisant $f(x \rhd y) = f(x) \rhd f(y)$, et un *morphisme de rack pointé* est un morphisme de rack f tel que $f(1) = 1$.

Le premier exemple de rack est celui de groupe. En effet, soit G un groupe, nous définissons un produit de rack $\rhd$ sur G en posant $g \rhd h = ghg^{-1}$ pour tout $g, h \in G$. Clairement, $g \rhd \_$ est une bijection avec inverse $g^{-1} \rhd \_$, et un calcul facile montre que l'identité de rack est satisfaite. Ainsi, nous avons un foncteur $Conj : Group \to Rack$. Ce foncteur a un adjoint à gauche $As : Rack \to Group$ défini sur les objets par $As(X) = F(X)/ < \{xyx^{-1}(x \rhd y) \,|\, x, y \in X\} >$ où $F(X)$ est le groupe libre engendré par X, et $< \{xyx^{-1}(x \rhd y) \,|\, x, y \in X\} >$ est le sous groupe normal engendré par $\{xyx^{-1}(x \rhd y) \,|\, x, y \in X\}$. Nous pouvons remarquer que $Conj(G)$ est un rack pointé. En effet, nous avons $1 \rhd g = g$ et $g \rhd 1 = 1$ pour tout $g \in G$, où 1 est l'élément neutre pour le produit de groupe. Ainsi, $Conj$ est un foncteur de $Group$ vers $PointedRack$. Ce foncteur a un adjoint à gauche $As_p : PointedRack \to Group$, défini sur les objets par $As_p(X) = As(X)/ < \{[1]\} >$, où $< \{[1]\} >$ est le sous groupe de $As(X)$ engendré par la classe $[1] \in As(X)$.

Un second exemple, et peut être le plus important, est celui de rack augmenté. Un *rack augmenté* est la donnée d'un groupe G, un G-ensemble X, et une application $X \xrightarrow{p} G$ satisfaisant l'*identité d'augmentation*, c'est à dire pour tout $g \in G$ et $x \in X$:

$$p(g.x) = gp(x)g^{-1}.$$

Alors nous définissons une structure de rack sur X en posant $x \rhd y = p(x).y$. Si il existe un élément $1 \in X$ tel que $p(1) = 1$ et $g.1 = 1$ pour tout $g \in G$, alors le rack augmenté $X \xrightarrow{p} G$ est dit *pointé*, et le rack associé $(X, \rhd)$ est pointé.

B.2.2 Cohomologie de rack pointé

Pour définir une théorie de cohomologie pour les racks, nous avons besoin d'une bonne notion de module de rack pointé. Dans cette article, nous prenons la définition donnée par N. Andruskiewitsch et M. Graña dans [AG03]. Soit X un rack pointé, un X-*module* est un groupe abélien A muni de deux familles d'homomorphismes $(\phi_{x,y})_{x,y \in X}$ et $(\psi_{x,y})_{x,y \in X}$ satisfaisant les axiomes sivants:

(M_0) $\phi_{x,y}$ est un isomorphisme.

(M_1) $\phi_{x,y \rhd z} \circ \phi_{y,z} = \phi_{x \rhd y, x \rhd z} \circ \phi_{x,z}$

(M_2) $\phi_{x,y \rhd z} \circ \psi_{y,z} = \psi_{x \rhd y, x \rhd z} \circ \phi_{x,y}$

(M_3) $\psi_{x,y \rhd z} = \phi_{x \rhd y, x \rhd z} \circ \psi_{x,z} + \psi_{x \rhd y, x \rhd z} \circ \psi_{x,y}$

(M_4) $\phi_{1,y} = id_A \quad \forall y \in X$ et $\psi_{x,1} = 0 \quad \forall x \in X$

Remarque B.2.1. Il existe une définition plus générale de module de rack (pointé) donné par N. Jackson dans [Jac07], mais nous n'avons pas besoin d'un tel degré de généralité. Cette définition de module de rack pointé coincide avec la définition de module de rack pointé homogène donné dans [Jac07].

Par exemple, il existe deux structures canoniques de X-module sur un $As_p(X)$-module. En effet, soit A un $As_p(X)$-module, c'est à dire un groupe abélien muni d'un morphisme de groupe $\rho : As_p(X) \to Aut(A)$, la première structure de X-module, appelée *symétrique*, que nous pouvons définir sur A est donné pour tout $x, y \in X$ par

$$\begin{aligned}
\phi_{x,y}(a) &= \rho_x(a)\\
\psi_{x,y}(a) &= a - \rho_{x \rhd y}(a).
\end{aligned}$$

La seconde, appelée *antisymétrique*, est donné pour tout $x, y \in X$ par

$$\begin{aligned}
\phi_{x,y}(a) &= \rho_x(a)\\
\psi_{x,y}(a) &= 0.
\end{aligned}$$

Avec cette définition de module, N. Andruskiewitsch et M. Graña définissent une théorie de cohomologie pour les racks pointés. Pour X un rack pointé et A un X-module, ils définissent un complexe de cochaine $\{CR^n(X, A), d_R^n\}_{n \in \mathbb{N}}$ en posant

$$CR^n(X, A) = \{f : X^n \to A \,|\, f(x_1, \ldots, 1, \ldots, x_n) = 0\}$$

and

$$\begin{aligned}
&d_R^n f(x_1, \ldots, x_{n+1}) =\\
&\sum_{i=1}^{n} (-1)^{i-1} \big(\phi_{x_1 \rhd \ldots \rhd x_i, x_1 \rhd \ldots \rhd \widehat{x_i} \rhd \ldots \rhd x_{n+1}}(f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1})) - f(x_1, \ldots, x_i \rhd x_{i+1}, \ldots, x_i \rhd x_{n+1})\big)\\
&\qquad + (-1)^n \psi_{x_1 \rhd \ldots \rhd x_n, x_1 \rhd \ldots \rhd x_{n-1} \rhd x_{n+1}}(f(x_1, \ldots, x_n))
\end{aligned}$$

Ce complexe est le même que celui défini dans [Jac07], mais dans le contexte des racks à gauche. En adaptant la preuve donnée par N. Jackson dans [Jac07], nous montrons que second groupe de cohomologie $HR^2(X, A)$ est en bijection avec l'ensemble des classes d'équivalences d'extensions abéliennes d'un rack pointé X par un X-module A. Une *extension abélienne* d'un rack pointé X par un X-module A étant un morphisme de rack pointé surjectif $E \overset{p}{\twoheadrightarrow} X$ qui satisfait les axiomes suivants

(E_0) pour tout $x \in X$, il existe une action à droite qui est libre et transitive de A sur $p^{-1}(x)$.

(E_1) pour tout $u \in p^{-1}(x), v \in p^{-1}(y), a \in A$, nous avons $(u.a) \rhd v = (u \rhd v).\psi_{x,y}(a)$.

(E_2) pour tout $u \in p^{-1}(x), v \in p^{-1}(y), a \in A$, nous avons $u \rhd (v.a) = (u \rhd v).\phi_{x,y}(a)$.

et deux extensions $E_1 \overset{p_1}{\twoheadrightarrow} X$, $E_2 \overset{p_2}{\twoheadrightarrow} X$ sont dites *equivalentes*, si il existe un isomorphisme de rack pointé $E_1 \xrightarrow{\theta} E_2$ qui satisfait les axiomes suivants

1. $p_2 \circ \theta = p_1$.
2. pour tout $x \in X, u \in p^{-1}(x), a \in A$, nous avons $\theta(u.a) = \theta(u).a$.

B.2.3 Racks de Lie

Pour généraliser les groupes de Lie, nous avons besoin de racks pointés munis d'une structure différentiable compatible avec la structure algébrique, c'est la notion de racks de Lie. Un *rack de Lie* est une variété differentiable X avec une structure de rack pointé telle que le produit $\rhd$ soit lisse, et telle que pour tout $x \in X$, c_x soit un diffeomorphisme. Nous verrons que l'espace tangent en l'élément neutre d'un rack de Lie est muni d'une structure d'algèbre de Leibniz.

Soit X un rack de Lie, un X-module A est dit *lisse* si A est un groupe de Lie abélien, et si $\phi : X \times X \times A \to A$ et $\psi : X \times X \times A \to A$ sont lisses. On peut alors définir une théorie de cohomologie pour les racks de Lie à valeurs dans un module lisse. Pour cela on définit un complexe de cochaînes $\{CR^n_s(X, A), d^n_R\}_{n\in\mathbb{N}}$ en prenant pour $CR^n_s(X, A)$, l'ensemble des fonctions $f : X^n \to A$ qui sont lisses dans un voisinage de $(1, \ldots, 1) \in X^n$ et telles que $f(x_1, \ldots, 1, \ldots, x_n) = 0$ pour tout $x_1, \ldots, x_n \in X$. La formule pour la différentielle d_R est la même que celle définie précedemment. Nous verrons qu'un cocycle (resp. cobord) de rack de Lie se dérive en un cocycle (resp. cobord) de Leibniz.

B.2.4 Rack local

Pour définir une structure d'algèbre de Lie sur l'espace tangent au neutre d'un groupe de Lie, on peut remarquer que l'on utilise uniquement la structure de groupe de Lie local au voisinage de 1. Nous verrons que cette remarque est toujours vrai pour les racks de Lie et algèbres de Leibniz.

Un *rack local* est un ensemble X muni d'un produit $\rhd$ défini sur un sous ensemble Ω de $X \times X$ à valeurs dans X, et tel que les axiomes suivants soient satisfaits:

1. Si $(x, y), (x, z), (y, z), (x, y\rhd z), (x\rhd y, x\rhd z) \in \Omega$, alors $x\rhd(y\rhd z) = (x\rhd y)\rhd(x\rhd z)$.
2. Si $(x, y), (x, z) \in \Omega$ et $x\rhd y = x\rhd z$, alors $y = z$.

Un rack local est dit *pointé* si il existe un élément $1 \in X$ tel que $1\rhd x$ et $x\rhd 1$ soient définis pour tout $x \in X$ et respectivement égaux à x et 1. Cet élément est alors appelé l'*élément neutre*. Un *rack de Lie local* est alors un rack local pointé $(X, \Omega, \rhd, 1)$ où X est une variété lisse, Ω est un sous ensemble ouvert de X et $\rhd : \Omega \to X$ est lisse. Par exemple dans un rack de Lie X, tout sous ensemble U contenant l'élément neutre est un rack de Lie local. Étant donné un tel rack de Lie local, on va définir un complexe de cochaînes associée.

Soit X un rack de Lie, U un sous ensemble de X contenant 1 et A un X-module lisse. On définit pour tout $n \in \mathbb{N}$, $CR^n_s(U, A)$ comme l'ensemble des applications $f : U_{n-loc} \to A$, lisses dans un voisinage du neutre, et telles que $f(x_1, \ldots, 1, \ldots, x_n) = 0$. Si A n'est pas antisymétrique, U_{n-loc} est le sous ensemble de $X \times U^{n-1}$ des n-uplets $(x_1, \ldots, x_n)$ qui vérifient $x_{i_1}\rhd \ldots \rhd x_{i_j} \in U$, pour tout $i_1 < \cdots < i_j, 2 \leq j \leq n$. Si A est antisymétrique, U_{n-loc} est le sous ensemble de $X^{n-1} \times U$ des n-uplets $(x_1, \ldots, x_n)$ qui vérifient $x_{i_1}\rhd \ldots \rhd x_{i_j}\rhd x_n \in U$, pour tout $i_1 < \cdots < i_j < n, 1 \leq j \leq n-1$. On vérifie facilement que la formule pour la différentielle d_R permet de définir un complexe de cochaînes $\{CR^n_s(U, A), d^n_R\}_{n\in\mathbb{N}}$. On définit alors la *cohomologie de rack de Lie U-local de X à coefficients dans A* comme la cohomologie du complexe de cochaînes $\{CR^n_s(U, A), d^n_R\}_{n\in\mathbb{N}}$.

B.3 Racks de Lie et algèbres de Leibniz

Dans cette section nous montrons l'existence d'une correspondance entre racks de Lie locaux et algèbres de Leibniz. Dans [Kin07], M.K. Kinyon montre que l'espace tangent en 1 à un rack de Lie X est muni d'une structure d'algèbre de Leibniz. Nous montrons alors facilement que ce résultat est encore vrai si l'on suppose que X est un rack de Lie local. Le principal théorème de cette thèse est la réciproque de ce résultat. C'est à dire que pour toute algèbre de Leibniz $\mathfrak{g}$, il existe un rack de Lie local X dont l'espace tangent en 1 est muni d'une structure d'algèbre de

Leibniz isomorphe à $\mathfrak{g}$. En fait nous montrons que le rack de Lie local intégrant $\mathfrak{g}$ est un rack de Lie local augmenté.

B.3.1 Des racks de Lie aux algèbres de Leibniz

Proposition B.3.1. *Soit X un rack de Lie, alors T_1X est une algebre de Leibniz.*

Preuve : Soit X un rack de Lie, on note par $\mathfrak{x}$ l'espace tangent de X en 1. La conjugaison induit $\forall x \in X$ un automorphisme de rack de Lie c_x. nous définissons $\forall x \in X$

$$Ad_x = T_1 c_x \in GL(\mathfrak{x})$$

et grâce à $c_{x \rhd y} = c_x \circ c_y \circ c_x^{-1}$ et $c_1 = id$ (cf. Définition B.2.1), ceci nous donne un morphisme de racks de Lie

$$Ad : X \to GL(\mathfrak{x})$$

Nous pouvons différentier Ad, et nous obtenons une application linéaire

$$ad : X \to \mathfrak{gl}(\mathfrak{x})$$

Alors nous définissons un crochet $[-,-]$ sur $\mathfrak{x} = T_1X$ en posant

$$[u,v] = ad(u)(v)$$

Maintenant, nous devons vérifier que ce crochet vérifit l'identité de Leibniz, c'est à dire

$$[u,[v,w]] = [[u,v],w] + [v,[u,w]]$$

Pour montrer cette identité, nous utilisons l'identité de rack

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$$

Soit $u, v, w \in \mathfrak{x}$ et γ_u (resp. γ_v and γ_w) un chemin lisse dans X, pointé en 1, tel que $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_u(s) = u$ (resp. $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_v(s) = v$ et $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_w(s) = w$). Nous avons

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0} (\gamma_u(r) \rhd (\gamma_v(s) \rhd \gamma_w(t))) &= \left.\frac{\partial}{\partial t}\right|_{t=0} ((c_{\gamma_u(r)} \circ c_{\gamma_v(s)})(\gamma_w(t)) \\
&= (Ad_{\gamma_u(r)} \circ Ad_{\gamma_v(s)})(w)
\end{aligned}$$

et

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0} ((\gamma_u(r) \rhd \gamma_v(s)) \rhd (\gamma_u(r) \rhd \gamma_w(t))) &= \left.\frac{\partial}{\partial t}\right|_{t=0} ((c_{\gamma_u(r) \rhd \gamma_v(s)} \circ c_{\gamma_u(r)})(\gamma_w(t)) \\
&= (Ad_{\gamma_u(r) \rhd \gamma_v(s)} \circ Ad_{\gamma_u(r)})(w) \\
&= (Ad_{c_{\gamma_u(r)}(\gamma_v(s))} \circ Ad_{\gamma_u(r)})(w)
\end{aligned}$$

De plus, si nous différentions par rapport à la variable s

$$\left.\frac{\partial}{\partial s}\right|_{s=0} ((Ad_{\gamma_u(r)} \circ Ad_{\gamma_v(s)})(w)) = (Ad_{\gamma_u(r)} \circ ad(v))(w)$$

et

$$\frac{\partial}{\partial s}\bigg|_{s=0}(Ad_{c_{\gamma_u(r)}(\gamma_v(s))}\circ Ad_{\gamma_u(r)})(w)=(ad(Ad_{\gamma_u(r)}(v))\circ Ad_{\gamma_u(r)})(w)$$

Et si nous différentions par rapport à la variable r, nous obtenons

$$\frac{\partial}{\partial t}\bigg|_{t=0}(Ad_{\gamma_u(r)}\circ ad(v))(w)=ad(u)(ad(v)(w))$$

et

$$\frac{\partial}{\partial r}\bigg|_{r=0}(ad(Ad_{\gamma_u(r)}(v))\circ Ad_{\gamma_u(r)})(w)=ad(ad(u)(v))(w)+ad(v)(ad(u)(w))$$

Ainsi, par identification, nous avons l'identité de Leibniz.

□

Par exemple, si G est un groupe de Lie nous obtenons la structure d'algèbre de Lie canonique sur T_1G. Si $X\xrightarrow{p}G$ est un rack de Lie augmenté, alors $T_1X\xrightarrow{T_1p}T_1G$ est une algèbre de Lie dans la catégorie des applications linéaires (cf. [LP98]). Cette structure induit alors une structure d'algèbre de Leibniz sur T_1X.

Nous pouvons remarquer qu'une structure lisse locale au voisinage de 1 est suffisante pour munir T_1X d'une structure d'algèbre de Leibniz.

Proposition B.3.2. *Soit X un rack de Lie local, alors T_1X est une algèbre de Leibniz.*

B.3.2 Des $As_p(X)$-modules aux représentations de Leibniz

Soit X un rack pointé. Un $As_p(X)$-module est un groupe abélien A muni d'un morphisme de groupes $\phi : As_p(X)\to Aut(A)$. Par adjonction, ceci est équivalent à la donnée d'un morphisme de rack pointé $\phi : X\to Conj(Aut(A))$.

Définition B.3.3. *Soit X un rack de Lie, un $As_p(X)$-**module lisse** est un $As_p(X)$-module A tel que*

1. *A est un groupe de Lie abélien.*
2. *$\phi : X\times A\to A$ est lisse.*

Rappellons que, étant donné une algèbre de Leibniz $\mathfrak{g}$, une $\mathfrak{g}$-représentation $\mathfrak{a}$ est un espace vectoriel muni de deux applications linéaires

$$[-,-]_L : \mathfrak{g}\otimes\mathfrak{a}\to\mathfrak{a}$$

et

$$[-,-]_R : \mathfrak{a}\otimes\mathfrak{g}\to\mathfrak{a}$$

satisfaisants les axiomes $(LLM),(LML)$ and (MLL). (cf. la sous section B.1.2).

Il y a deux classes particulières de représentations de Leibniz. La première, que nous avons appelé *symétrique*, est constituée des représentations où $[-,-]_L=-[-,-]_R$. la seconde, que nous avons appelé *antisymétrique*, est constituée des représentations où $[-,-]_R=0$. Étant donnée une algèbre de Leibniz $\mathfrak{g}$ et $\mathfrak{a}$ un espace vectoriel muni d'un morphisme d'algèbres de

Leibniz $\phi : \mathfrak{g} \to End(\mathfrak{a})$, nous pouvons définir deux structures de $\mathfrak{g}$-representation de Leibniz sur $\mathfrak{a}$. Une est *symétrique* et définie par

$$[x,a]_L = \phi_x(a) \text{ and } [a,x]_R = -\phi_x(a), \quad \forall x \in \mathfrak{g}, a \in \mathfrak{a}.$$

et l'autre est *antisymétrique* et définie par

$$[x,a]_L = \phi_x(a) \text{ and } [a,x]_R = 0, \quad \forall x \in \mathfrak{g}, a \in \mathfrak{a}.$$

De plus, dans la sous section B.2.2 nous avons vu que étant donné un rack (de Lie) X et A un $As_p(X)$-module lisse, nous pouvons définir deux structures de X-modules (lisses) sur A. Une est appelée *symétrique* et définie par

$$\phi_{x,y}(a) = \phi_x(a) \text{ and } \psi_{x,y}(a) = a - \phi_{x \rhd y}(a), \quad \forall x,y \in X, a \in A.$$

et l'autre est appelé *antisymétrique* et définie par

$$\phi_{x,y}(a) = \phi_x(a) \text{ and } \psi_{x,y}(a) = 0, \quad \forall x,y \in X, a \in \mathcal{A}.$$

Ces deux constructions, l'une dans le monde des algèbres de Leibniz et l'autre dans le monde des racks de Lie, sont similaires car la première est la version infinitésimale de l'autre. En effet, soit (A,ϕ,ψ) un X-module symétrique lisse, nous avons par définition deux applications lisses

$$\phi : X \times X \times A \to A \text{ and } \psi : X \times X \times A \to A$$

avec $\phi_{1,1} = id, \psi_{1,1} = 0$. Ainsi les differentielles des deux applications au point $(1,1)$ nous donnent deux applications

$$\epsilon : X \times X \to Aut(\mathfrak{a}); \epsilon(x,y) = T_1\phi_{x,y}$$

et

$$\chi : X \times X \to End(\mathfrak{a}); \chi(x,y) = T_1\psi_{x,y}$$

Ces applications sont lisses, donc nous pouvons les dériver au point $(1,1)$. Nous obtenons alors deux applications

$$T_{(1,1)}\epsilon : \mathfrak{x} \oplus \mathfrak{x} \to End(\mathfrak{a})$$

et

$$T_{(1,1)}\chi : \mathfrak{x} \oplus \mathfrak{x} \to End(\mathfrak{a})$$

On pose alors

$$[-,-]_L : \mathfrak{g} \otimes \mathfrak{a} \to \mathfrak{a}; [u,m]_L = T_{(1,1)}\epsilon(u,0)(m)$$

et

$$[-,-]_R : \mathfrak{a} \otimes \mathfrak{g} \to \mathfrak{a}; [m,u]_R = T_{(1,1)}\chi(0,u)(m)$$

Lemma B.3.4. *L'application linéaire $[-,-]_L$ satisfait l'axiome (LLM).*

Preuve : nous voulons montrer que $\forall u,v \in \mathfrak{x}, m \in \mathfrak{a}$

$$[u,[v,m]_L]_L = [[u,v],m]_L + [v,[u,m]_L]_L$$

par hypothèse, ϕ satisfait la relation suivante $\forall x,y,z \in X$

$$\phi_{x,y \rhd z} \circ \phi_{y,z} = \phi_{x \rhd y, x \rhd z} \circ \phi_{x,z}$$

et si on prend $z = 1$, nous obtenons

$$\phi_{x,1} \circ \phi_{y,1} = \phi_{x \rhd y,1} \circ \phi_{x,1}$$

ainsi

$$T_1\phi_{x,1} \circ T_1\phi_{y,1} = T_1\phi_{x \rhd y,1} \circ T_1\phi_{x,1}$$

c'est à dire

$$\epsilon(x,1) \circ \epsilon(y,1) = \epsilon(x \rhd y, 1) \circ \epsilon(x,1)$$

Soit $u, v \in \mathfrak{x}$ and γ_u (resp. γ_v) un chemin dans X pointé en 1 tel que $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_u(s) = u$ (resp. $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_v(s) = v$). Nous avons

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0} (\epsilon(\gamma_u(s),1) \circ \epsilon(\gamma_v(t),1)) &= \epsilon(\gamma_u(s),1) \circ \left.\frac{\partial}{\partial t}\right|_{t=0} \epsilon(\gamma_v(t),1) \\
&= \epsilon(\gamma_u(s),1) \circ [v,-]_L
\end{aligned}$$

et

$$\begin{aligned}
\left.\frac{\partial}{\partial s}\right|_{s=0} (\epsilon(\gamma_u(s),1) \circ [v,0]_L) &= (\left.\frac{\partial}{\partial s}\right|_{s=0} \epsilon(\gamma_u(s),1)) \circ [v,-]_L \\
&= [u,-]_L \circ [v,-]_L \\
&= [u,[v,-]_L]_L
\end{aligned}$$

D'un autre côté, nous avons

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (\epsilon(\gamma_u(s) \rhd \gamma_v(t),1) \circ \epsilon(\gamma_u(s),1)) = [Ad_{\gamma_u(s)}(v),-]_L \circ \epsilon(\gamma_u(s),1)$$

et

$$\begin{aligned}
\left.\frac{\partial}{\partial s}\right|_{s=0} ([Ad_{\gamma_u(s)}(v),-]_L \circ \epsilon(\gamma_u(s),1)) &= \left.\frac{\partial}{\partial s}\right|_{s=0} ([Ad_{\gamma_u(s)}(v),-]_L) \circ \epsilon(1,1) \\
&\quad + [Ad_{\gamma_u(0)}(v),-]_L \circ \left.\frac{\partial}{\partial s}\right|_{s=0} (\epsilon(\gamma_u(s),1)) \\
&= [[u,v],-]_L + [v,-]_L \circ [u,-]_L \\
&= [[u,v],-]_L + [v,[u,-]]_L
\end{aligned}$$

Ainsi par identification

$$[u,[v,-]_L]_L = [[u,v],-]_L + [v,[u,-]]_L$$

C'est à dire, $[-,-]_L$ satisfait (LLM).

□

Lemma B.3.5. *Si (A,ϕ,ψ) est symétrique, alors $[-,-]_L = -[-,-]_R$ et si (A,ϕ,ψ) est anti-symétrique, alors $[-,-]_R = 0$.*

Preuve : Supposons que (A, ϕ, ψ) est antisymétrique, alors $\psi = 0$, et il est clair que $[-,-]_R = 0$. Maintenant, supposons que (A, ϕ, ψ) est symétrique, c'est à dire $\psi_{x,y}(a) = a - \phi_{x \rhd y,1}(a)$. Nous avons $\chi(x,y) = T_1\psi_{x,y} = id - \epsilon(x \rhd y, 1)$. Soit $u \in \mathfrak{x}$ et γ_u un chemin dans X pointé en 1 tel que $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_u(s) = u$. Nous avons

$$\begin{aligned}
[-,u]_R &= T_{(1,1)}\chi(0,u) \\
&= \left.\frac{\partial}{\partial s}\right|_{s=0} \chi(1, \gamma_u(s)) \\
&= \left.\frac{\partial}{\partial s}\right|_{s=0} (id - \epsilon(\gamma_u(s), 1)) \\
&= -\left.\frac{\partial}{\partial s}\right|_{s=0} \epsilon(\gamma_u(s), 1) \\
&= -[u,-]_L
\end{aligned}$$

□

Finalement, nous avons montré la proposition suivante:

Proposition B.3.6. *Soit X un rack de Lie, $\mathfrak{x}$ son algebre de Leibniz, A un groupe de Lie abélien et $\mathfrak{a}$ son algèbre de Lie. Si (A, ϕ, ψ) est un X-module lisse symétrique (resp. antisymétrique), alors $(\mathfrak{a}, [-,-]_L, [-,-]_R)$ est un $\mathfrak{x}$-module symétrique (resp. antisymétrique).*

B.3.3 De la cohomologie de rack de Lie à la cohomologie de Leibniz

Proposition B.3.7. *Soit X un rack de Lie et A un $As(X)$-module lisse. Il existe des morphismes de complexes de cochaînes*

$$CR^n_p(X, A^s)_s \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^s)$$

et

$$CR^n_p(X, A^a)_s \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^a)$$

donnés par $\delta^n(f)(a_1, \ldots, a_n) = d^n f(1, \ldots, 1)((a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_n))$ (où $d^n f$ est la différentielle n-ième de f).

Preuve : Soit $f \in CR^n_p(X, A^s)$ et $(x_0, \ldots, x_n) \in X^{n+1}$. Nous avons

$$d_R f(x_0, \ldots, x_n) = \sum_{i=1}^{n+1} (-1)^{i-1} \phi_{x_1 \rhd \ldots \rhd x_i}(f(x_1, \ldots, \hat{x_i}, \ldots, x_{n+1})) - f(x_1, \ldots, x_i \rhd x_{i+1}, \ldots, x_i \rhd x_{n+1})$$

(Ici $\phi_{x_1 \rhd \ldots \rhd x_i} = \phi_{x_1 \rhd \ldots \rhd x_i, x_1 \rhd \ldots \rhd \widehat{x_1} \rhd \ldots \rhd x_{n+1}}$)
Soit $(\gamma_0(t_0), \ldots, \gamma_n(t_n))$ une famille de chemins $\gamma_i :]-\epsilon_i, +\epsilon_i[\to V$ tels que

$$\gamma_i(0) = 1 \text{ et } \frac{\partial}{\partial s}_{s=0} \gamma_i(s) = x_i$$

Nous voulons montrer que

$$\delta^{n+1}(d^n_R f) = d^n_L(\delta^n(f))$$

Nous avons par définition

$$\delta^{n+1}(d^n_R f)(a_0, \ldots, a_n) = \left.\frac{\partial^{n+1}}{\partial t_0 \ldots \partial t_n}\right|_{t_i=0} df(\gamma_0(t_0), \ldots, \gamma_n(t_n))$$

Lemma B.3.8. $\forall i \in \{1, \dots, n\}$

$$\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{|t_i=0} \phi_{\gamma_0(t_0)\triangleright\dots\triangleright\gamma_i(t_i)}(f(\gamma_0(t_0), \dots, \gamma_{i-1}(t_{i-1}), \gamma_{i+1}(t_{i+1}), \dots, \gamma_n(t_n))) = a_i.D_n(f)(a_0, \dots, \widehat{a_i}, \dots, a_n)$$

Preuve :
Notation:

$$(t_0, \dots, t_i) \mapsto \phi_{\gamma_0(t_0)\triangleright\dots\triangleright\gamma_i(t_i)}$$

sera noté par

$$(t_0, \dots, t_i) \mapsto A(t_0, \dots, t_i)$$

et

$$(t_0, \dots, \widehat{t_i}, \dots, t_n) \mapsto f(\gamma_0(t_0), \dots, \gamma_{i-1}(t_{i-1}), \gamma_{i+1}(t_{i+1}), \dots, \gamma_n(t_n))$$

sera noté par

$$(t_0, \dots, \widehat{t_i}, \dots, t_n) \mapsto m(t_0, \dots, \widehat{t_i}, \dots, t_n)$$

Nous avons

$$\begin{aligned}
\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_k=0} A(t_0, \dots, t_i)(m(t_0, \dots, t_n)) &= \frac{\partial^{n}}{\partial t_0 \dots \partial t_{n-1}}\Big|_{t_k=0} (\frac{\partial}{\partial t_n}_{t_n=0} A(t_0, \dots, t_i)(m(t_0, \dots, 0)) \\
&+ \frac{\partial^{n}}{\partial t_0 \dots \partial t_{n-1}}\Big|_{t_k=0} (A(t_0, \dots, t_i)(\frac{\partial}{\partial t_n}\Big|_{t_n=0} m(t_0, \dots, t_n))) \\
&= \frac{\partial^{n}}{\partial t_0 \dots \partial t_{n-1}}\Big|_{t_k=0} (A(t_0, \dots, t_i)(\frac{\partial}{\partial t_n}_{t_n=0} m(t_0, \dots, t_n)))
\end{aligned}$$

car $f(x_0, \dots, 1, \dots, x_n) = 0$.
Et par le même argument nous avons

$$\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_k=0} A(t_0, \dots, t_i)(m(t_0, \dots, t_n)) = \frac{\partial}{\partial t_i}\Big|_{t_i=0} A(0, \dots, 0, t_i)(\frac{\partial^{n}}{\partial t_0 \dots \partial t_n}\Big|_{t_k=0} m(t_0, \dots, t_n))$$

Ainsi

$$\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_i=0} A(t_0, \dots, t_n)(m(t_0, \dots, t_n)) = x_i.D_n(f)(x_0, \dots, \hat{x_i}, \dots, x_n)$$

□

Lemma B.3.9. $\forall i \in \{1, \dots, n\}$

$$\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_i=0} f(\gamma_0(t_0), \dots, \gamma_i(t_i)\triangleright\gamma_{i+1}(t_{i+1}), \dots, \gamma_i(t_i)\triangleright\gamma_n(t_n)) = \sum_{k=i+1}^{n} \delta n f(a_0, \dots, [a_i, a_k], \dots, a_n)$$

Preuve : Nous avons $\frac{\partial^{n+1}}{\partial t_0 \dots \partial t_n}\Big|_{t_i=0} f(\gamma_0(t_0), \dots, \gamma_i(t_i)\triangleright\gamma_{i+1}(t_{i+1}), \dots, \gamma_i(t_i)\triangleright\gamma_n(t_n))$ qui est égal à

$$\frac{\partial}{\partial t_i}\Big|_{t_i=0} d^n f(1, \dots, 1)((a_0, 0, \dots, 0), \dots, (0, \dots, Ad_{\gamma_i(t_i)}(a_{i+1}), \dots, 0), \dots, (0, \dots, 0, Ad_{\gamma_i(t_i)}(a_n))$$

De plus, ceci est égal à

$$\sum_{k=i+1}^{n} d^n f(1, \dots, 1)((a_0, 0, \dots, 0), \dots, (0, \dots, [a_i, a_k], \dots, 0), \dots, (0, \dots, 0, a_n))$$

et cette expresion est égale à $\displaystyle\sum_{k=i+1}^{n} \delta^n f(a_0, \ldots, [a_i, a_k], \ldots, a_n)$.

$\square$

Donc

$$\begin{aligned}\delta^{n+1}(d_R^n f)(a_0, \ldots, a_n) &= \sum_{i=0}^{n}(-1)^i \Big(a_i.\delta^n(f)(a_0, \ldots, \hat{a_i}, \ldots, a_n) - \sum_{k=i+1}^{n} \delta^n f(a_0, \ldots, [a_i, a_k], \ldots, a_n)\Big) \\ &= \sum_{i=0}^{n}(-1)^i a_i.\delta^n(f)(a_0, \ldots, \hat{a_i}, \ldots, a_n) + \sum_{0\leq i<k\leq n}(-1)^{i+1}\delta^n f(a_0, \ldots, [a_i, a_k], \ldots, a_n)\end{aligned}$$

c'est à dire

$$\delta^{n+1}(d_R^n f) = d_L^n(\delta^n(f))$$

La preuve est exactement la même pour le cas où A est antisymétrique.

$\square$

Nous pouvons remarquer que nous n'avons besoin que d'une identité de cocycle local au voisinage de 1. Ainsi nous avons

Proposition B.3.10. *Soit X un rack de Lie, U un voisinage de 1 dans X et A un $As(X)$-module lisse. Nous avons des morphismes de complexes de cochaînes*

$$CR_p^n(U, A^s) \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^s)$$

et

$$CR_p^n(U, A^a) \xrightarrow{\delta^n} CL^n(\mathfrak{x}, \mathfrak{a}^a)$$

donnés par $\delta^n(f)(a_0, \ldots, a_n) = d^n f(1, \ldots, 1)((a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_n))$.

B.3.4 De la cohomologie de Leibniz vers la cohomologie de rack de Lie local

Dans cette section, nous étudions deux cas d'intégration de cocycles de Leibniz. Cette section sera utilisée dans la section suivante pour intégrer toutes algèbre de Leibniz en rack de Lie local augmenté.

Premièrement, nous étudions l'intégration d'un 1-cocycle de Leibniz appartenant à $ZL^1(\mathfrak{g}, \mathfrak{a}^s)$ en un 1-cocycle de rack de Lie appartenant à $ZR_p^1(G, \mathfrak{a}^s)_s$, où G est un groupe de Lie simplement connexe d'algèbre de Lie $\mathfrak{g}$ et $\mathfrak{a}$ une représentation de G.

Deuxièmement, nous utilisons le résultat de la première partie pour étudier l'intégration d'un 2-cocycle de Leibniz appartenant à $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ en un 2-cocycle de rack de Lie local appartenant à $ZR_p^2(U, \mathfrak{a}^a)_s$, où U est un voisinage de 1 dans un groupe de Lie simplement connexe G d'algèbre de Lie $\mathfrak{g}$, et $\mathfrak{a}$ une représentation de G. C'est la seconde partie que nous utiliserons pour intégrer les algèbres de Leibniz.

Des 1-cocycles de Leibniz aux 1-cocycles de rack de Lie

Soit G un groupe de Lie simplement connexe et $\mathfrak{a}$ une représentaiton de G. Nous voulons définir un morphisme I^1 de $ZL^1(\mathfrak{g}, \mathfrak{a}^s)$ vers $ZR_p^1(G, \mathfrak{a}^s)_s$ qui envoit $BL^1(\mathfrak{g}, \mathfrak{a}^s)$ dans $BR_p^1(G, \mathfrak{a}^s)_s$. Pour ceci, nous posons

$$I^1(\omega)(g) = \int_{\gamma_g} \omega^{eq}$$

où $\omega \in ZL^1(\mathfrak{g}, \mathfrak{a}^s)$, $\gamma : G \times [0,1] \to G$ est une application lisse telle que γ_g soit un chemin de 1 à g, γ_1 est le chemin constant égal à 1, et ω^{eq} est la forme différentielle equivariante à gauche fermée appartenant à $\Omega^1(G, \mathfrak{a})$ définie par

$$\omega^{eq}(g)(m) = g.(\omega(T_g L_{g^{-1}}(m)))$$

Par définition, il est clair que $I^1(\omega)(1) = 0$.

Pour le moment, $I^1(\omega)$ dépend de γ, mais du fait que ω soit un cocycle, la dépendance par rapport à γ disparait.

Proposition B.3.11. *I^1 ne dépend pas de γ.*

Proof : Soit $\gamma, \gamma' : G \times [0,1] \to G$ tel que $\gamma_g(0) = \gamma'_g(0) = 1$ et $\gamma_g(1) = \gamma'_g(1) = g$. Nous allons montrer que

$$\int_{\gamma_g} \omega^{eq} = \int_{\gamma'_g} \omega^{eq}$$

Nous avons $\int_{\gamma_g} \omega^{eq} - \int_{\gamma'_g} \omega^{eq} = \int_{\gamma_g - \gamma'_g} \omega^{eq}$. Comme $H_1(G) = 0$ et $\partial(\gamma_g - \gamma'_g) = 0$, il existe $\sigma_g : [0,1]^2 \to G$ tel que $\gamma_g - \gamma'_g = \partial \sigma_g$. Donc

$$\begin{aligned} \int_{\gamma_g} \omega^{eq} - \int_{\gamma'_g} \omega^{eq} &= \int_{\gamma_g - \gamma'_g} \omega^{eq} \\ &= \int_{\partial \sigma_g} \omega^{eq} \\ &= \int_{\sigma_g} d_{dR} \omega^{eq} \\ &= 0 \end{aligned}$$

Ainsi I^1 ne dépend pas de γ.

$\square$

Proposition B.3.12. *I^1 envoit les cocycles sur les cocycles et les cobords sur les cobords.*

Preuve : Premièrement, soit $\omega \in ZL^1(\mathfrak{g}, \mathfrak{a}^s)$, nous avons

$$\begin{aligned}
d_R I(\omega)(g,h) &= g.I(\omega)(h) - I(\omega)(g \triangleright h) - (g \triangleright h).I(\omega)(g) + I(\omega)(g) \\
&= g.\int_{\gamma_h} \omega^{eq} - \int_{\gamma_{g\triangleright h}} \omega^{eq} - (g\triangleright h).\int_{\gamma_g} \omega^{eq} + \int_{\gamma_g} \omega^{eq} \\
&= \int_{\gamma_h} g.\omega^{eq} - \int_{\gamma_{g\triangleright h}} \omega^{eq} - \int_{\gamma_g} (g\triangleright h).\omega^{eq} + \int_{\gamma_g} \omega^{eq} \\
&= \int_{g\gamma_h} \omega^{eq} - \int_{\gamma_{g\triangleright h}} \omega^{eq} - \int_{(g\triangleright h)\gamma_g} \omega^{eq} + \int_{\gamma_g} \omega^{eq} \\
&= \int_{g\gamma_h - \gamma_{g\triangleright h} - (g\triangleright h)\gamma_g + \gamma_g} \omega^{eq}
\end{aligned}$$

Comme $H^1(G) = 0$ et $\partial(g\gamma_h - \gamma_{g\triangleright h} - (g\triangleright h)\gamma_g + \gamma_g) = 0$, il existe $\gamma_{g,h} : [0,1]^2 \to G$ such that $\partial\gamma_{g,h} = g\gamma_h - \gamma_{g\triangleright h} - (g\triangleright h)\gamma_g + \gamma_g$. Ainsi, nous avons

$$\begin{aligned}
d_R I(\omega)(g,h) &= \int_{\partial\gamma_{g,h}} \omega^{eq} \\
&= \int_{\gamma_{g,h}} d_{dR}\omega^{eq} \\
&= 0
\end{aligned}$$

Ainsi $ZL^1(\mathfrak{g}, \mathfrak{a}^s)$ est envoyé dans $ZR^1_p(G, \mathfrak{a}^s)_s$.

Deuxièmement, soit $\omega \in BL^1(\mathfrak{g}, \mathfrak{a}^s)$. Il existe $\beta \in \mathfrak{a}$ tel que $\omega(m) = m.\beta$. Nous avons

$$\begin{aligned}
I(\omega)(g) &= \int_{\gamma_g} \omega^{eq} \\
&= \int_{\gamma_g} (d_L\beta)^{eq} \\
&= \int_{\gamma_g} d_{dR}\beta^{eq} \\
&= \beta^{eq}(g) - \beta^{eq}(1) \\
&= g.\beta - \beta \\
&= d_R\beta(g)
\end{aligned}$$

Ainsi $BL^1(\mathfrak{g}, \mathfrak{a}^s)$ est envoyé dans $BR^1_p(G, \mathfrak{a}^s)_s$.

□

Proposition B.3.13. *I^1 est un inverse à gauche de δ^1.*

Preuve : Soit $\omega \in ZL^1(\mathfrak{g}, \mathfrak{a}^s)$. Soit $\varphi : U \to \mathfrak{g}$ une carte locale contenant 1 tel que $\varphi(1) = 0$ et $d\varphi^{-1}(0) = id$.
Nous définissons pour $x \in \mathfrak{g}$ l'application lisse $\alpha_x :]-\epsilon, +\epsilon[\to U$ en posant

$$\alpha_x(s) = \varphi^{-1}(sx)$$

et nous définissons, $\forall s \in]-\epsilon, +\epsilon[$, l'application lisse $\gamma_{\alpha_x(s)} : [0,1] \to U$ en posant

$$\gamma_{\alpha_x(s)}(t) = \varphi^{-1}(tsx)$$

Nous avons

$$\begin{aligned}
\delta^1(I^1(\omega))(x) &= \frac{\partial}{\partial s}\Big|_{s=0} I^1(\omega)(\alpha_x(s)) \\
&= \frac{\partial}{\partial s}\Big|_{s=0} \int_{\gamma_{\alpha_x(s)}} \omega^{eq} \\
&= \frac{\partial}{\partial s}\Big|_{s=0} \int_{[0,1]} \gamma^*_{\alpha_x(s)} \omega^{eq} \\
&= \frac{\partial}{\partial s}\Big|_{s=0} \int_{[0,1]} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_{\alpha_x(s)}(t))dt
\end{aligned}$$

De plus, $\frac{\partial}{\partial t}\Big|_{t=0} \gamma_{\alpha_x(s)}(t) = \frac{\partial}{\partial t}\Big|_{t=0} \varphi^{-1}(stx) = sx$, donc

$$\begin{aligned}
\delta^1(I^1(\omega))(x) &= \frac{\partial}{\partial s}\Big|_{s=0} \int_{[0,1]} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(sx)dt \\
&= \int_{[0,1]} \frac{\partial}{\partial s}\Big|_{s=0} \omega^{eq}(\gamma_{\alpha_x(s)}(t))(sx)dt \\
&= \int_{[0,1]} \frac{\partial}{\partial s}\Big|_{s=0} \omega^{eq}(\varphi^{-1}(tsx))(sx)dt \\
&= \int_{[0,1]} \frac{\partial}{\partial s}\Big|_{s=0} (\varphi^{-1})^*\omega^{eq}(tsx)(sx)dt \\
&= \int_{[0,1]} \frac{\partial}{\partial s}\Big|_{s=0} s(\varphi^{-1})^*\omega^{eq}(tsx)(x)dt \\
&= \int_{[0,1]} (\varphi^{-1})^*\omega^{eq}(0)(x)dt \\
&= \omega(x) \int_{[0,1]} dt \\
&= \omega(x)
\end{aligned}$$

Ainsi $\delta^1 \circ I^1 = id$

$\square$

Remarque B.3.14. On peut montrer qu'il existe un isomorphisme $H^1(\mathfrak{g}, \mathfrak{a}) \overset{D^1}{\to} H^1_s(G, \mathfrak{a})$ de la cohomologie d'algèbre de Lie vers la cohomologie de groupes de Lie (cf. [Nee04]). Nous pouvons remarquer que I^1 est la même application que celle définie dans [Nee04] pour prouver la surjectivité. En fait, pour prouver la Proposition B.3.12, nous avons juste combiner le résultat de K.H. Neeb et le fait que l'on a une injection $H^1_s(G, \mathfrak{a})_s \overset{id}{\hookrightarrow} HR^1_s(G, \mathfrak{a}^s)$.

Des 2-cocycles de Leibniz aux 2-cocycles de rack de Lie local

Soit G un groupe de Lie simplement connexe, U un voisinage de 1 dans G tel que l'application log soit définie sur U et $\mathfrak{a}$ une représentation de G. Dans la proposition (3.3.4) nous avons défini pour tout $n \in \mathbb{N}$ les applications

$$HR^n_s(U, \mathfrak{a}^a) \xrightarrow{[\delta^n]} HL^n(\mathfrak{g}, \mathfrak{a}^a)$$

Dans la prochaine section, nous verrons qu'une algèbre de Leibniz peut être intégré en un rack de Lie local puisque $[\delta^2]$ est surjectif. Plus précisemment, si nous pouvons construire un inverse à gauche pour $[\delta^2]$, alors il nous donne une méthode explicite pour construire un rack de Lie local qui intègre l'algèbre de Leibniz.

Dans cette section, nous définissons un morphisme $[I^2]$ de $HL^2(\mathfrak{g}, \mathfrak{a}^a)$ vers $HR^2_s(U, \mathfrak{a}^a)$, et nous montrons que c'est un inverse à gauche pour $[\delta^2]$. Pour construire l'application $[I^2]$, nous adaptons une méthode d'intégration des cocycles d'algèbre de Lie en cocycles de groupes de Lie par intégration sur des simplex. Cette méthode est due à W.T. Van Est ([Est54]) et utilisée par K.H. Neeb ([Nee02, Nee04]) pour le cas de la dimension infinie.

Définition de I^2

Nous voulons définir une application de $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ vers $ZR^2_p(U, \mathfrak{a}^a)_s$ telle que $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ soit envoyé sur $BR^2_p(U, \mathfrak{a}^a)_s$. Dans la section précédente, nous avons intégré un 1-cocycle de Leibniz sur une algebre de Lie $\mathfrak{g}$ à coefficients dans un module symétrique $\mathfrak{a}^s$. Dans la Proposition (B.1.1), nous avons expliciter un morphisme entre $CL^2(\mathfrak{g}, \mathfrak{a}^a)$ et $CL^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s)$, qui envoit $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ dans $ZL^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s)$ et $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ dans $BL^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s)$. Ainsi, nous pouvons définir une application

$$I : ZL^2(\mathfrak{g}, \mathfrak{a}^a) \to ZR^1_p(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$$

qui envoit $BL^2(\mathfrak{g}, \mathfrak{a}^a)$ dans $BR^1_p(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$. C'est la composée

$$ZL^2(\mathfrak{g}, \mathfrak{a}^a) \xrightarrow{\tau^2} ZL^1(\mathfrak{g}, Hom(\mathfrak{g}, \mathfrak{a})^s) \xrightarrow{I^1} ZR^1_p(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$$

Maintenant, nous voulons définir une application de $ZR^1_p(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$ vers $ZR^2_p(U, \mathfrak{a}^a)$. Soit $\beta \in CR^1_p(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$, β a ces valeurs dans la représentations $Hom(\mathfrak{g}, \mathfrak{a})$, donc pour tout $g \in G$, nous pouvons considérer la forme différentielle equivariante $\beta(g)^{eq} \in \Omega^1(G, \mathfrak{a})$ définie par

$$\beta(g)^{eq}(h)(m) := h.(\beta(g)(T_h L_{h^{-1}}(m)))$$

Alors nous définissons un élement dans $CR^2_p(U, \mathfrak{a}^a)$ en posant

$$f(g,h) = \int_{\gamma_{g \rhd h}} (\beta(g))^{eq}$$

où $\gamma : G \times [0,1] \to G$ est une application lisse telle que pour tout $g \in G$, γ_g est un chemin de 1 à g dans G et $\gamma_1 = 1$.
Pour l'instant, un élément de $ZR^1_p(G, Hom(\mathfrak{g}, \mathfrak{a})^s)_s$ n'est pas nécéssairement envoyé sur un élément de $ZR^2_p(U, \mathfrak{a}^a)_s$. Pour atteindre cet objectif, nous devons spécifier l'application γ, et nous la définissons en posant

$$\gamma_g(s) = exp(s \log(g))$$

Alors, nous définissons $I^2 : ZL^2(\mathfrak{g}, \mathfrak{a}^a) \to CR^2_p(U, \mathfrak{a}^a)_s$ en posant $\forall (g,h) \in U_{2-loc}$

$$I^2(\omega)(g,h) = \int_{\gamma_{g \rhd h}} (I(\omega)(g))^{eq}$$

Par définition, il est clair que $I^2(\omega)(g,1) = I^2(\omega)(1,g) = 0$.

Propriétés de I^2

Proposition B.3.15. *I^2 envoit $ZL^2(\mathfrak{g},\mathfrak{a}^a)$ dans $ZR^2_p(U,\mathfrak{a}^a)_s$.*

Pour prouver cette proposition, nous avons besoins des lemmes suivants.

Lemma B.3.16. *Pour tout $(g,h)\in U_{2-loc}$, nous avons $\gamma_{g\rhd h}=g\rhd\gamma_h$.*

Preuve : Soit $(g,h)\in U_{2-loc}$, par définition nous avons

$$\gamma_{g\rhd h}(s)=exp(s(\log(g\rhd h)))$$

Par naturalité de l'exponentielle, et a fortiori du logarithme, nous avons

$$\begin{aligned}
exp(s(log(g\rhd h))) &= exp(sAd_g(\log(h)))\\
&= exp(Ad_g(s\log(h)))\\
&= c_g(exp(s\log(h)))\\
&= c_g(\gamma_h(s))
\end{aligned}$$

Ainsi $\gamma_{g\rhd h}=g\rhd\gamma_h$.

□

Lemma B.3.17. *Soit G un groupe de Lie, $\mathfrak{a}$ une représentation de G et $\omega\in Hom(\mathfrak{g},\mathfrak{a})$. Nous avons $\forall g\in G$*

$$g.(\omega^{eq})=c_g^*((g.\omega)^{eq})$$

Preuve : Sot $g,h\in G, x\in T_hG$, nous avons:

$$\begin{aligned}
(c_g^*((g.\omega)^{eq}))(h)(x) &= (g.\omega)^{eq}(g\rhd h)(d_hc_g(x))\\
&= (g\rhd h).(g.\omega)(d_{g\rhd h}L_{g\rhd h^{-1}}(d_hc_g(x)))\\
&= (g\rhd h).(g.\omega)(d_h(c_g\circ L_{h^{-1}})(x))\\
&= (g\rhd h).(g.(\omega(Ad(g^{-1})(d_h(c_g\circ L_{h^{-1}})(x)))))\\
&= gh.(\omega(d_h(c_{g^{-1}}\circ c_g\circ L_{h^{-1}})(x)))\\
&= g.(h.(\omega(d_hL_{h^{-1}}(x))))\\
&= g.(\omega^{eq}(h)(x))
\end{aligned}$$

Ainsi $c_g^*((g.\omega)^{eq})=g.(\omega)^{eq}$.

□

Preuve de la proposition : Soit $\omega\in ZL^2(\mathfrak{g},\mathfrak{a}^a)$ et $(g,h,k)\in U_{3-loc}$. Nous avons

$$\begin{aligned}
d_R(I^2(\omega))(g,h,k) &= g.I^2(\omega)(h,k)-I^2(\omega)(g\rhd h,g\rhd k)-(g\rhd h).I^2(\omega)(g,k)+I^2(\omega)(g,h\rhd k)\\
&= g.\int_{\gamma_{h\rhd k}}(I(\omega)(h))^{eq}-\int_{\gamma_{g\rhd(h\rhd k)}}(I(\omega)(g\rhd h))^{eq}-(g\rhd h).\int_{\gamma_{g\rhd k}}(I(\omega)(g))^{eq}\\
&+\int_{\gamma_{g\rhd(h\rhd k)}}(I(\omega)(g))^{eq}\\
&= \int_{\gamma_{h\rhd k}}g.((I(\omega)(h))^{eq})-\int_{\gamma_{g\rhd(h\rhd k)}}(I(\omega)(g\rhd h))^{eq}-\int_{\gamma_{g\rhd k}}(g\rhd h).((I(\omega)(g))^{eq})\\
&+\int_{\gamma_{g\rhd(h\rhd k)}}(I(\omega)(g))^{eq}
\end{aligned}$$

Grâce au Lemma B.3.17, nous avons

$$\begin{aligned}
d_R(I^2(\omega))(g,h,k) &= \int_{\gamma_{h\rhd k}} c_g^*((g.I(\omega)(h))^{eq}) - \int_{\gamma_{g\rhd(h\rhd k)}} I(\omega)(g\rhd h)^{eq} - \int_{\gamma_{g\rhd k}} c_{g\rhd h}^*(((g\rhd h).I(\omega)(g))^{eq}) \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} I(\omega)(g)^{eq} \\
&= \int_{c_g\circ\gamma_{h\rhd k}} (g.I(\omega)(h)^{eq} - \int_{\gamma_{g\rhd(h\rhd k)}} I(\omega)(g\rhd h)^{eq} - \int_{c_{g\rhd h}\circ\gamma_{g\rhd k}} ((g\rhd h).I(\omega)(g))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} I(\omega)(g)^{eq}
\end{aligned}$$

et grâce au Lemma B.3.16, nous avons

$$\begin{aligned}
d_R(I^2(\omega))(g,h,k) &= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g\rhd(h\rhd k)}} I(\omega)(g\rhd h)^{eq} - \int_{\gamma_{g\rhd(h\rhd k}} ((g\rhd h).I(\omega)(g))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} I(\omega)(g)^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h))^{eq} - I(\omega)(g\rhd h)^{eq} - ((g\rhd h).I(\omega)(g))^{eq} + I(\omega)(g)^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h) - I(\omega((g\rhd h) - (g\rhd h).I(\omega)(g) + I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} d_R(I(\omega))(g,h) \\
&= 0
\end{aligned}$$

Ainsi $ZL^2(\mathfrak{g},\mathfrak{a}^a)$ est envoyé dans $ZR_p^2(U,\mathfrak{a}^a)_s$.

$\square$

Proposition B.3.18. *I^2 envoit $BL^2(\mathfrak{g},\mathfrak{a}^a)$ dans $BR_p^2(U,\mathfrak{a}^a)_s$*

Preuve : Soit $\omega \in BL^2(\mathfrak{g},\mathfrak{a}^a)$, il existe un élément $\beta \in CL^1(\mathfrak{g},\mathfrak{a}^a)$ tel que $\omega = d_L\beta$. Nous avons

$$\begin{aligned}
I(\omega)(g) &= I^1(\tau^2(\omega))(g) \\
&= \int_{\gamma_g} (\tau^2(\omega))^{eq} \\
&= \int_{\gamma_g} (\tau^2(d_L\beta))^{eq}
\end{aligned}$$

Le fait que $\{\tau^n\}_{n\in\mathbb{N}}$ est un morphisme de complexe de cochaînes implique que

$$\begin{aligned}
I(\omega)(g) &= \int_{\gamma_g} (d(\tau^1(\beta)))^{eq} \\
&= \int_{\gamma_g} d_{dR}((\tau^1(\beta))^{eq}) \\
&= \int_{\partial\gamma_g} (\tau^1(\beta))^{eq} \\
&= g.\beta - \beta
\end{aligned}$$

Ainisi pour $(g,h) \in U_{2-loc}$ nous avons en utilisant le Lemma B.3.17

$$\begin{aligned}
I_2(\omega)(g,h) &= \int_{\gamma_{g\rhd h}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g\rhd h}} ((g.\beta)-\beta)^{eq} \\
&= \int_{\gamma_{g\rhd h}} (g.\beta)^{eq} - \int_{\gamma_{g\rhd h}} \beta^{eq} \\
&= \int_{\gamma_{g\rhd h}} (c^*_{g^{-1}}(g.(\beta^{eq}))) - \int_{\gamma_{g\rhd h}} \beta^{eq} \\
&= \int_{c_{g^{-1}}\circ\gamma_{g\rhd h}} g.(\beta^{eq}) - \int_{\gamma_{g\rhd h}} \beta^{eq} \\
&= g.\int_{c_{g^{-1}}\circ\gamma_{g\rhd h}} \beta^{eq} - \int_{\gamma_{g\rhd h}} \beta^{eq} \\
&= g.\int_{\gamma_h} \beta^{eq} - \int_{\gamma_{g\rhd h}} \beta^{eq} \\
&= d_R(I^1(\beta))(g,h)
\end{aligned}$$

Ainsi $BL^2(\mathfrak{g},\mathfrak{a}^a)$ est envoyé dans $BR^2_p(U,\mathfrak{a})_s$.

□

Proposition B.3.19. *I^2 est un inverse à gauche pour δ^2.*

Preuve : Soit $x,y \in \mathfrak{g}$, et I_x (resp. I_y) un intervalle dans $\mathbb{R}$ tel que $\epsilon_x(s) = exp(sx)$ (resp. $\epsilon_y(s) = exp(sy)$) soit défini $\forall s \in I_x$ (resp. $\forall s \in I_y$). L'application $\epsilon_x \rhd \epsilon_y : I_x \times I_y \to G$ est continue, donc il existe W un sous ensemble ouvert de $I_x \times I_y$ tel que $(\epsilon_x \rhd \epsilon_y)(W) \subseteq U$. Ainsi, il existe un intervalle $J \subseteq I_x \cap I_y$ tel que $\epsilon_x(s) \rhd \epsilon_y(t) \in U \quad \forall (s,t) \in J \times J$.

Nous devons montrer

$$\delta^2 \circ I^2 = id$$

Soit $\omega \in ZL^2(\mathfrak{g},\mathfrak{a}^a)$. Par définition, nous avons

$$\begin{aligned}
\delta^2(I^2(\omega))(x,y) &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{|s,t=0} I^2(\omega)(\epsilon_x(s),\epsilon_y(s)) \\
&= \left.\frac{\partial^2}{\partial s \partial t}\right|_{|s,t=0} \int_{\gamma_{\epsilon_x(s)\rhd\epsilon_y(t)}} (I(\omega)(\epsilon_x(s)))^{eq} \\
&= \left.\frac{\partial}{\partial s}\right|_{|s=0} \left(\left.\frac{\partial}{\partial t}\right|_{|t=0} \int_{\gamma_{\epsilon_y(t)}} c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s)))^{eq} \right)
\end{aligned}$$

Premièrement, nous calculons

$$\left.\frac{\partial}{\partial t}\right|_{|t=0} \int_{\gamma_{\epsilon_y(t)}} c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s)))^{eq}$$

Par souci de clarté, nous posons $\alpha = c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s)))^{eq}$ and $\beta_t = \gamma_{\epsilon_y(t)}$. Nous avons

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{|t=0} \int_\beta \alpha &= \frac{\partial}{\partial t}\Big|_{|t=0} \int_{[0,1]} \beta^* \alpha \\
&= \frac{\partial}{\partial t}\Big|_{|t=0} \int_{[0,1]} f_t(r) dr \\
&= \int_{[0,1]} \frac{\partial}{\partial t}\Big|_{|t=0} f_t(r) dr
\end{aligned}$$

où $f_t(r) = \alpha(\beta_t(r))(\beta_t'(r))$.
Nous avons

$$\frac{\partial}{\partial t}\Big|_{|t=0} f_t(r) = (\frac{\partial}{\partial t}\Big|_{|t=0} \alpha(\beta_t(r)))\beta_0'(r) + (\alpha(\beta_0(r)))(\frac{\partial}{\partial t}\Big|_{|t=0} \beta_t'(r))$$

De plus, nous avons

$$\alpha(\beta_0(r)) = \alpha(1)$$

$$\beta_0'(r) = 0$$

et

$$\frac{\partial}{\partial t}\Big|_{|t=0} \beta_t'(r) = y$$

Donc nous avons

$$\frac{\partial}{\partial t}\Big|_{|t=0} \int_\beta \alpha = \int_{[0,1]} \alpha(1)(x) dr = \alpha(1)(y)$$

et

$$\delta^2(I^2(\omega))(x, y) = \frac{\partial}{\partial s}\Big|_{|s=0} (c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s)))^{eq})(1)(y)$$

nous avons de plus

$$\begin{aligned}
c^*_{\epsilon_x(s)}(I(\omega)(\epsilon_x(s))^{eq})(1)(y) &= (I(\omega)(\epsilon_x(s)))^{eq}(c_{\epsilon_x(s)}(1))(Ad_{\epsilon_x(s)}(y)) \\
&= I(\omega)(\epsilon_x(s))(Ad_{\epsilon_x(s)}(y)) \\
&= (\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y))
\end{aligned}$$

Si l'on pose $\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq} = \sigma(s)$ and $Ad_{\epsilon_x(s)}(y) = \lambda(s)$, nous avons

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_{|s=0} ((\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y))) &= \frac{\partial}{\partial s}\Big|_{|s=0} \sigma(s)(\lambda(s)) \\
&= \sigma'(0)(\lambda(0)) + \sigma(0)(\lambda'(0))
\end{aligned}$$

Nous avons

$$\sigma(0) = 0$$

$$\lambda(0) = y$$

et

$$\sigma'(0) = \tau^2(\omega)(x)$$

Donc

$$\frac{\partial}{\partial s}\Big|_{|s=0}((\int_{\gamma_{\epsilon_x(s)}} \tau^2(\omega)^{eq})(Ad_{\epsilon_x(s)}(y))) = \tau^2(\omega)(x)(y)$$

Ainsi $\delta^2(I^2(\omega))(x,y) = \omega(x,y)$.

$\square$

Remarque B.3.20. Supposons que nous ayons un 2-cocycle de Leibniz ω qui soit aussi un 2-cocycle de Lie. Dans ce cas, nous pouvons intégrer ω cocycle de rack de Lie local, mais aussi en un cocycle de groupe de Lie local (cf. Lemma 5.2 p.18 dans [Nee04]). Il est alors naturel de se demander si les deux constructions sont reliées l'une à l'autre.

Proposition B.3.21. *Soit G un groupe de Lie, $\mathfrak{g}$ son algèbre de Lie, $\mathfrak{a}$ une représentation de G, $\omega \in \Lambda^2(\mathfrak{g}, \mathfrak{a})$ et γ_1, γ_2 des chemins lisses de G pointé en 1. Alors*

$$\int_{\gamma_1} \big(\int_{\gamma_2} (\tau^2(\omega))^{eq}\big)^{eq} = \int_{\gamma_1\gamma_2} \omega^{eq}$$

où $\gamma_1\gamma_2 : [0,1]^2 \to G; (s,t) \mapsto \gamma_1(t)\gamma_2(s)$.

Preuve : D'une part, nous avons

$$\begin{aligned}
\int_{\gamma_1\gamma_2} \omega^{eq} &= \int_{[0,1]^2} (\gamma_1\gamma_2)^*\omega^{eq} \\
&= \int_{[0,1]^2} \omega^{eq}(\gamma_1\gamma_2)(\frac{\partial}{\partial s}\gamma_1(t)\gamma_2(s), \frac{\partial}{\partial t}\gamma_1(t)\gamma_2(s))dsdt
\end{aligned}$$

et cette expression est égale à

$$\int_{[0,1]^2} \gamma_1(t)\gamma_2(s).\omega\big(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)), Ad_{\gamma_2(s)^{-1}}(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))\big) \tag{B.4}$$

D'autre part, nous avons

$$\begin{aligned}
\int_{\gamma_1} \big(\int_{\gamma_2} (\tau^2(\omega))^{eq}\big)^{eq} &= \int_{[0,1]} \gamma_1^*(\int_{\gamma_2} (\tau^2(\omega)^{eq})^{eq}) \\
&= \int_{[0,1]} (\int_{\gamma_2} (\tau^2(\omega))^{eq})^{eq}(\gamma_1(t))(\frac{\partial}{\partial t}\gamma_1(t))dt \\
&= \int_{[0,1]} \gamma_1(t).(\int_{\gamma_2} (\tau^2(\omega))^{eq})(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt \\
&= \int_{[0,1]} \gamma_1(t).(\int_{[0,1]} (\tau^2(\omega))^{eq}(\gamma_2(s))(\frac{\partial}{\partial s}\gamma_2(s)))(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt
\end{aligned}$$

Cette expression est égale à

$$\int_{[0,1]} \gamma_1(t).(\int_{[0,1]} \gamma_2(s).(\tau^2(\omega))(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)))ds)(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt$$

qui est égale à

$$\int_{[0,1]} \gamma_1(t).(\int_{[0,1]} \gamma_2(s).\omega(d_{\gamma_2(s)}L_{\gamma_2(s)^{-1}}(\frac{\partial}{\partial s}\gamma_2(s)), Ad_{\gamma_2(s)^{-1}}(\cdot))ds)(d_{\gamma_1(t)}L_{\gamma_1(t)^{-1}}(\frac{\partial}{\partial t}\gamma_1(t)))dt$$

En utilisant le théorème de Fubini, nous montrons alors que cette expression est égale à (B.4).

□

Si nous appliquons ce résultat au cas où $\gamma_1(s) = \gamma_{g\rhd h}(s) = exp(s\log(g\rhd h))$ et $\gamma_2(s) = \gamma_g(s) = exp(s\log(g))$ pour $(g,h) \in U_{2-loc}$, nous obtenons alors

Corollaire B.3.22. *Si $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a) \cap Z^2(\mathfrak{g}, \mathfrak{a})$, alors $I^2(\omega)(g,h) = \iota^2(\omega)(g,h) - \iota^2(\omega)(g\rhd h, g)$ où $\iota^2(\omega)(g,h) = \int_{\sigma_{g,h}} \omega^{eq}$ est un 2-cocycle de groupe de Lie local (cf. Lemma 5.2 p.18 dans [Nee04]).*

Nous pouvons remarquer que I^2 est plus qu'un cocycle de rack de Lie local. Précisement, si ω est dans $ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ alors l'identité de cocycle de rack de Lie local satisfaite par $I^2(\omega)$, provient d'une autre identité satisfaite pas $I^2(\omega)$. En effet, l'application I^2 est définie en utilisant I, et pour vérifier que I^2 envoit les cocycles de Leibinz dans les cocycles de rack local, nous avons utilisé la Proposition B.3.12. Cette proposition établit que I^1 envoit les cocycles de Lie dans les cocycles de rack. Mais nous avons remarqué dans la Remarque B.3.14 que l'identité de cocycle de rack satisfaite par $I^1(\omega)$, provient de l'identité de cocycle de groupe. Ainsi, il est naturel de penser que nous avons oublier de la structure sur $I^2(\omega)$. La proposition suivante explique quelle est l'identité satisfaite par $I^2(\omega)$ induisant l'identité de rack local.

Proposition B.3.23. *Si $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a)$, alors $I^2(\omega)$ satisfait l'identité*

$$g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g, h\rhd k) = 0, \quad \forall (g,h,k) \in U_{3-loc}$$

De plus, cette identité induit l'identité de cocycle de rack local.

Preuve : Soit $\omega \in ZL^2(\mathfrak{g}, \mathfrak{a}^a)$ et $(g,h,k) \in U_{3-loc}$ nous avons:

$$\begin{aligned} g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g,h\rhd k) &= g.\int_{\gamma_{h\rhd k}} (I(\omega)(h))^{eq} - \int_{\gamma_{(gh)\rhd k}} (I(\omega)(gh))^{eq} \\ &+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \\ &= \int_{\gamma_{h\rhd k}} g.((I(\omega)(h))^{eq}) - \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(gh))^{eq} \\ &+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \\ &= \int_{\gamma_{h\rhd k}} c_g^*((g.I(\omega)(h))^{eq}) - \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(gh))^{eq} \\ &+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \end{aligned}$$

$$
\begin{aligned}
g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g,h\rhd k) &= \int_{c_g\circ\gamma_{h\rhd k}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(gh))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h))^{eq} - \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(gh))^{eq} \\
&+ \int_{\gamma_{g\rhd(h\rhd k)}} (I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h))^{eq} - (I(\omega)(gh))^{eq} + I(\omega)(g)^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} (g.I(\omega)(h) - I(\omega)(gh) + I(\omega)(g))^{eq} \\
&= \int_{\gamma_{g\rhd(h\rhd k)}} d(I(\omega))(g,h) \\
&= 0
\end{aligned}
$$

Ainsi $I^2(\omega)$ satisfait l'identité voulue.
De plus, soit $(g,h,k) \in U_{3-loc}$, nous avons

$$
\begin{aligned}
d_R(I^2(\omega))(g,h,k) &= g.(I^2(\omega)(h,k)) - I^2(\omega)(g\rhd h, g\rhd k) - (g\rhd h).I^2(\omega)(g,k) + I^2(\omega)(g,h\rhd k) \\
&= g.(I^2(\omega)(h,k)) - I^2(\omega)(g\rhd h, g\rhd k) - (g\rhd h).I^2(\omega)(g,k) + I^2(\omega)(g,h\rhd k) \\
&- I^2(\omega)(gh,k) + I^2(\omega)(gh,k) \\
&= \big(g.(I^2(\omega)(h,k)) - I^2(\omega)(gh,k) + I^2(\omega)(g,h\rhd k)\big) \\
&- \big((g\rhd h).I^2(\omega)(g,k) - I^2(\omega)(gh,k) + I^2(\omega)(g\rhd h, g\rhd k)\big) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

Ainsi $I^2(\omega)$ est un cocycle de rack de Lie local.

□

Nous verrons dans la prochaine section que cette identité permet d'intégrer toutes algèbres de Leibniz en un rack de Lie local augmenté.

B.3.5 Des algèbres de Leibniz aux racks de Lie locaux

Dans cette section, nous exposons le théorème principale de notre thèse. Dans la Proposition B.3.1 nous avons vu que l'espace tangent en 1 d'un rack de Lie (local) est muni d'une structure d'algèbre de Leibniz. Réciproquement, nous montrons que toutes algèbres de Leibniz peut être intégré en un rack de Lie local augmenté. Notre construction est explicite, et par cette construction, une algèbre de Lie est intégré en un groupe de Lie. Réciproquement, nous montrons qu'un rack de Lie local augmenté dont l'espace tangent en 1 est une algèbre de Lie est nécéssairement un groupe de Lie (local). C'est à dire qu'il existe une structure de groupe de Lie sur le rack de Lie local augmenté, et la conjugaison sur le rack de Lie local augmenté est la conjugaison du groupe.

L'idée de la preuve est simple et utilise la connaissance du troisième théorème de Lie. Soit $\mathfrak{g}$ une algèbre de Leibniz. Tout d'abord, nous décomposons l'espace vectoriel $\mathfrak{g}$ en la somme directe d'algèbres de Leibniz $\mathfrak{g}_0$ et $\mathfrak{a}$ que nous savons intégrer. Comme nous connaissons le théorème pour les algèbres de Lie, c'est le cas si $\mathfrak{g}$ est une extension abélienne d'une algèbre de Lie $\mathfrak{g}_0$ par une $\mathfrak{g}_0$-representation $\mathfrak{a}$. Ainsi, $\mathfrak{g}$ est isomorphe à $\mathfrak{a} \oplus_\omega \mathfrak{g}_0$. l'algèbre de Leibniz $\mathfrak{a}$ est abélienne, donc s'intègre en $\mathfrak{a}$, et $\mathfrak{g}_0$ est une algèbre de Lie, donc s'intègre en un groupe de Lie simplement connexe G_0. Maintenant, nous devons comprendre comment recoller $\mathfrak{a}$ et G_0. C'est à dire, nous devons comprendre comment la donnée du recollement ω s'intègre en en une donnée de recollement f entre $\mathfrak{a}$ et G_0. C'est le cocycle de rack de Lie local $I^2(\omega)$, construit dans la section précédente, qui répond à cette question. Ainsi, nous avons montré qu'une algèbre de Leibniz $\mathfrak{g}$ s'intègre en un rack de Lie local de la forme $\mathfrak{a} \times_f G_0$.

Soit $\mathfrak{g}$ une algèbre de Leibniz, il y a plusieurs façons de voir $\mathfrak{g}$ comme une extension abélienne d'une algèbre de Lie $\mathfrak{g}_0$ par une $\mathfrak{g}_0$-représentation $\mathfrak{a}$. Ici, nous prenons l'extension abélienne associée au centre (à gauche) de $\mathfrak{g}$. Par définition, le centre (à gauche) est

$$Z_L(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [x, y] = 0 \ \ \forall y \in \mathfrak{g}\}$$

$Z_L(\mathfrak{g})$ est un idéal de $\mathfrak{g}$ et nous pouvons considérer le quotient de $\mathfrak{g}$ par $Z_L(\mathfrak{g})$. C'est une algèbre de Leibniz, et plus précisemment une algèbre de Lie car $\mathfrak{g}_{ann}$ est inclu dans $Z_L(\mathfrak{g})$. Nous notons ce quotient par $\mathfrak{g}_0$. Ainsi, pour une algèbre de Leibniz $\mathfrak{g}$, il existe une extension abélienne canonique donnée par

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$$

Cette extension donne une structure de $\mathfrak{g}_0$-représentation à $Z_L(\mathfrak{g})$, et par définition de $Z_L(\mathfrak{g})$, cette représentation est antisymétrique. Cette classe d'équivalence est caractérisée par une classe de cohomologie dans $HL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$. Ainsi il existe un élément $\omega \in ZL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$ tel que l'extension $Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$ soit équivalente à

$$Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g}_0 \oplus_\omega Z_L(\mathfrak{g}) \overset{\pi}{\twoheadrightarrow} \mathfrak{g}_0$$

$\mathfrak{g}_0$ est une algèbre de Lie, donc s'intègre en un groupe de Lie connexe et simplement connexe G_0, et $Z_L(\mathfrak{g})$ est une algèbre de Lie abélienne, donc s'intègre en elle même. $Z_L(\mathfrak{g})$ est une $\mathfrak{g}_0$-représentation de Lie et G_0 est simplement connexe, donc par le second théorème de Lie $Z_L(\mathfrak{g})$ est un G_0-module lisse. Ainsi $Z_L(\mathfrak{g})$ est muni d'une structure de G_0-module (de rack) antisymétrique lisse. Le cocycle $\omega \in ZL^2(\mathfrak{g}, Z_L(\mathfrak{g}))$ s'intègre en un cocycle de rack de Lie local $I^2(\omega) \in ZR^2_p(G_0, Z_L(\mathfrak{g}))_s$, et nous pouvons poser une structure de rack de Lie local sur le produit cartésien $G_0 \times Z_L(\mathfrak{g})$ en posant

$$(g, a) \rhd (h, b) = (g \rhd h, \phi_{g,h}(b) + \psi_{g,h}(a) + I^2(\omega)(g, h))$$

où $\phi_{g,h}(b) = g.b$ et $\psi_{g,h}(a) = 0$. C'est à dire nous avons

$$(g, a) \rhd (h, b) = (g \rhd h, g.b + I^2(\omega)(g, h)))$$

Par construction il est clair que ce rack de Lie local a son espace tangent en 1 muni d'une structure d'algèbre de Leibniz isomorphe à $\mathfrak{g}$. Finalement, nous avons montré la proposition suivante

Théorème B.3.24. *Toute algèbre de Leibniz $\mathfrak{g}$ s'intègre en un rack de Lie local de la forme*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a$$

avec pour conjugaison

$$(g,a)\triangleright(h,b) = (g\triangleright h, g.b + I^2(\omega)(g,h)) \tag{B.5}$$

et élément neutre $(1,0)$*, où* G_0 *est une groupe de Lie,* $\mathfrak{a}$ *un* G_0*-module et* $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$*. Réciproquement, l'espace tangent en* 1 *d'un rack de Lie local de cette forme est muni d'une structure d'algèbre de Leibniz.*

Nous demandons plus dans notre problème de départ. En effet, nous demandons en plus qu'une algèbre de Lie s'intègre en un groupe de Lie. Ainsi, quand $\mathfrak{g}$ est une algèbre de Lie, nous devons montrer que $G_0 \times Z_L(\mathfrak{g})$ est muni d'une structure de groupe de Lie telle que la conjugaison sur $G_0 \times_{I^2(\omega)} Z_L(\mathfrak{g})$ soit induit par le produit de rack sur $Conj(G_0 \times Z_L(\mathfrak{g}))$.

Soit $\mathfrak{g}$ une algèbre de Lie, le centre à gauche $Z_L(\mathfrak{g})$ est égal au centre $Z(\mathfrak{g})$. L'extension abélienne $Z_L(\mathfrak{g}) \overset{i}{\hookrightarrow} \mathfrak{g} \overset{p}{\twoheadrightarrow} \mathfrak{g}_0$ munit $Z_L(\mathfrak{g})$ avec une structure antisymétrique mais aussi une structure symétrique, donc une structure triviale. Cette extension devient une extension centrale et le cocycle $\omega \in ZL^2(\mathfrak{g}_0, Z(\mathfrak{g}))$ est aussi dans $Z^2(\mathfrak{g}_0, Z(\mathfrak{g}))$. D'une part, avec ω nous pouvons construire un cocycle de rack de Lie local $I^2(\omega)$, et d'autre part, nous pouvons construire un cocycle de groupe de Lie $\iota^2(\omega)$. In (3.4.12), nous avons montré que $I^2(\omega)(g,h) = \iota^2(\omega)(g,h) - \iota^2(\omega)(g\triangleright h, g)$ pour tout $(g,h) \in U_{2_{loc}}$. Ainsi, la conjugaison dans $G_0 \times_{I^2(\omega)} Z(\mathfrak{g})$ peut être écrite

$$(g,a)\triangleright(h,b) = (g\triangleright h, I^2(\omega)(g,h)) = (g\triangleright h, \iota^2(\omega)(g,h) - \iota^2(\omega)(g\triangleright h, g)).$$

Cette formule est celle de la conjugaison dans le groupe $G_0 \times_{\iota^2(\omega)} Z(\mathfrak{g})$, où le produit est défini par

$$(g,a)(h,b) = (gh, \iota^2(g,h))$$

Réciproquement, supposons qu'un rack de Lie de la forme $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ a son espace tangent en 1, $\mathfrak{g}_0 \oplus_\omega \mathfrak{a}^a$, muni d'une structure d'algèbre de Lie. Nécessairement, $\mathfrak{a}$ est une $\mathfrak{g}_0$-representation triviale et $\omega \in Z^2(\mathfrak{g}_0, \mathfrak{a})$. Ainsi, comme avant, $I^2(\omega)(g,h) = \iota^2(\omega)(g,h) - \iota^2(\omega)(g\triangleright h, g)$ pour tout $(g,h) \in U_{2-loc}$ et la conjugaison définie par (B.5) est induite par la conjugaison venant de la structure de groupe sur $G_0 \times_{\iota^2(\omega)} \mathfrak{a}$. Finalement, nous avons le théorème suivant

Théorème B.3.25. *Toute algèbre de Leibniz* $\mathfrak{g}$ *s'intègre en un rack de Lie local de la forme*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a$$

avec pour conjugaison

$$(g,a)\triangleright(h,b) = (g\triangleright h, g.b + I^2(\omega)(g,h)) \tag{B.6}$$

et élément neutre $(1,0)$*, où* G_0 *est un groupe de Lie,* $\mathfrak{a}$ *une représentation de* G_0 *et* $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$*. Réciproquement, l'espace tangent en* 1 *d'un rack de Lie local de cette forme est muni d'une structure d'algèbre de Leibniz.*

De plus, dans le cas spécial où $\mathfrak{g}$ *est une algèbre de Lie, la construction ci-dessus munie* $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ *d'un produit de rack provenant de la conjugaison dans un groupe de Lie. Réciproquement, si l'espace tangent en* 1 *de* $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ *est une algèbre de Lie, alors* $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ *peut être muni d'une structure de groupe de Lie, et la conjugaison induite par la structure de groupe de Lie est celle définie par* (B.6)*.*

Des algèbres de Leibniz aux rack de Lie augmentés locaux

Soit $\mathfrak{g}_0$ une algèbre de Lie, $\mathfrak{a}$ une $\mathfrak{g}$-représentation et $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$. Dans (3.4.5), nous avons montré que $I^2(\omega)$ est un cocycle de rack de Lie local. Nous avons aussi montré qu'il satisfait l'identité

$$g.I^2(\omega)(h,k) - I^2(\omega)(gh,k) + I^2(\omega)(g, h \rhd k) = 0 \tag{B.7}$$

for all $(g,h,k) \in U_{3-loc}$.

La question naturelle est alors : Quelle structure algébrique sur $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ est encodé par cette identité?

Définition B.3.26. *Soit G un groupe. Un G-**ensemble local** est un ensemble X muni d'une application ρ définie sur un sous ensemble Ω de $G \times X$ à valeurs dans X telle que les axiomes suivants soient satisfaits*

1. *Si $(h,x), (gh,x), (g, \rho(h,x)) \in \Omega$, alors $\rho(g, \rho(h,x)) = \rho(gh, x)$.*
2. *$\forall x \in X, (1,x) \in \Omega$ et $\rho(1,x) = x$.*

*Un G-**ensemble lisse** est une variété lisse X munie d'une structure de G-ensemble local vérifiant*

1. *Ω est un sous ensemble ouvert de $X \times X$.*
2. *$\rho : \Omega \to X$ est lisse.*

*Un **point fixe** est un élément $x_0 \in X$ tel que $\forall g \in G, (g, x_0) \in \Omega$ et $\rho(g, x_0) = x_0$.*

Dans la proposition suivante, nous montrons que l'identité (B.7) munit $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ d'une structure de G_0-ensemble local.

Proposition B.3.27. *$G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ est un G_0-ensemble local lisse, et $(1,0)$ est un point fixe.*

Preuve : Nous définissons un sous ensemble ouvert Ω et une application lisse ρ par

1. $\Omega = \{(g,(h,b)) \in G_0 \times (G_0 \times_{I^2(\omega)} \mathfrak{a}^a) | (g,h) \in U_{2-loc}\}$.
2. $\rho(g,(h,b)) = (g \rhd h, g.b + I^2(\omega)(g,h))$.

Soit $(h,(k,z)), (gh,(k,z)), (g, \rho(h,(k,z))) \in \Omega$. Ceci est équivalent à la condition $(h,k), (gh,k), (g, h \rhd k) \in U_{2-loc}$, c'est à dire $(g,h,k) \in U_{3-loc}$. Nous avons

$$\begin{aligned}\rho(g, \rho(h,(k,z))) &= \rho(g, (h \rhd k, h.z + I^2(\omega)(h,k))) \\ &= (g \rhd (h \rhd k), g.(h.z) + g.I^2(\omega)(h,k) + I^2(\omega)(g, h \rhd k))\end{aligned}$$

En utilisant les identités (2) et $(gh) \rhd k = g \rhd (h \rhd k)$, nous avons

$$\begin{aligned}\rho(g, \rho(h,(k,z))) &= ((gh) \rhd k, (gh).z + I^2(\omega)(gh,k)) \\ &= \rho(gh, \rho(k,z))\end{aligned}$$

Donc $\rho(g, \rho(h,(k,z))) = \rho(gh, \rho(k,z))$.
De plus, nous avons $\rho(1,(k,z)) = (1 \rhd k, 1.z + I^2(\omega)(1,k)) = (k,z)$ et $\rho(g,(1,0)) = (g \rhd 1, g.0 + I^2(\omega)(g,1)) = (1,0)$. Ainsi $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ est un G_0-ensemble local lisse et $(1,0)$ est un point fixe pour cette action local.

□

Nous pouvons remarquer que l'on peut reconstruire le produit de rack sur $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ à partir de la formule de l'action de G_0. En effet, nous avons

$$(g,a) \rhd (h,b) = g.(h,b) = p(g,a).(h,b)$$

où p est la projection sur le premier facteur $G_0 \times_{I^2(\omega)} \mathfrak{a}^a \overset{p}{\twoheadrightarrow} G_0$. De plus, $p(1,0) = 1$ et p est équivariant. En effet, soit $(g,(h,y)) \in \Omega$, nous avons $p(\rho(g.(h,y))) = p(g \rhd h, g.y + I^2(\omega)(g,h)) = g \rhd h = g.p(h,y)$. Ainsi nous avons montré la proposition suivante

Proposition B.3.28. *$G_0 \times_{I^2(\omega)} \mathfrak{a}^a \overset{p}{\twoheadrightarrow} G_0$ est un rack de Lie augmenté local.*

Ainsi nous pouvons réécrire notre théorème principal

Théorème B.3.29. *Toute algèbre de Leibniz $\mathfrak{g}$ s'intègre en un rack de Lie augmenté local de la forme*

$$G_0 \times_{I^2(\omega)} \mathfrak{a}^a \overset{p}{\twoheadrightarrow} G_0$$

avec action local

$$g.(h,b) = (g \rhd h, g.b + I^2(\omega)(g,h))$$

et élément neutre $(1,0)$, où G_0 est un groupe de Lie, $\mathfrak{a}$ une représentation de G_0 et $\omega \in ZL^2(\mathfrak{g}_0, \mathfrak{a}^a)$. Réciproquement, l'espace tangent en 1 d'un rack de Lie augmenté local de cette forme est muni d'une structure d'algèbre de Leibniz.

De plus, dans le cas spécial où $\mathfrak{g}$ est une algèbre de Lie, la construction ci-dessus munie $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ d'un produit de rack provenant de la conjugaison dans un groupe de Lie. Réciproquement, si l'espace tangent en 1 de $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ est une algèbre de Lie, alors $G_0 \times_{I^2(\omega)} \mathfrak{a}^a$ peut être muni d'une structure de groupe de Lie, et la conjugaison induite par la structure de groupe de Lie est celle définie par (B.6).

B.3.6 Exemples d'intégration d'algèbres de Leibniz non scindées

En dimension 4

Exemple B.3.30. Soit $\mathfrak{g} = \mathbb{R}^4$. Nous définissons une application linéaire $\mathfrak{g}$ par

$$\begin{aligned}
[e_1, e_1] &= e_4 \\
[e_1, e_2] &= e_4 \\
[e_2, e_1] &= -e_4 \\
[e_2, e_2] &= e_4 \\
[e_3, e_3] &= e_4
\end{aligned}$$

et tous les crochets des autres combinaisons d'éléments de la base égaux à zéro. Nous avons

$$[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)] = (0, 0, 0, x_1y_1 + x_1y_2 - x_2y_1 + x_2y_2 + x_3y_3)$$

Proposition B.3.31. $(\mathfrak{g}, [-,-])$ *est une algèbre de Leibniz.*

Preuve : Nous avons

$$[(x_1, x_2, x_3, x_4), [(y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, 0)$$

et

$$[(y_1, y_2, y_3, y_4), [(x_1, x_2, x_3, x_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, 0)$$

et

$$[[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)], (z_1, z_2, z_3, z_4)] = (0, 0, 0, 0)$$

Ainsi le crochet $[-, -]$ satisfait l'identité de Leibniz.

$\square$

Pour suivre la construction expliquée ci-dessus, nous devons déterminer le centre à gauche $Z_L(\mathfrak{g})$, le quotient de $\mathfrak{g}$ par $Z_L(\mathfrak{g})$ noté $\mathfrak{g}_0$, l'action de $\mathfrak{g}_0$ sur $Z_L(\mathfrak{g})$ et le 2-cocycle de Leibniz décrivant l'extension abélienne $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Soit $x \in Z_L(\mathfrak{g})$, pour $y = (1, 0, 0, 0), y = (0, 1, 0, 0)$ ou $y = (0, 0, 1, 0)$ dans $\mathfrak{g}$, nous avons $[x, y] = 0$. Ceci implique que $x_1 = x_2 = x_3 = 0$. Réciproquement, tout élément dans $\mathfrak{g}$ ayant les trois premières coordonnées égales à 0 est dans $Z_L(\mathfrak{g})$. Ainsi $Z_L(\mathfrak{g}) =< e_4 >$ et $\mathfrak{g}_0 \simeq< e_1, e_2, e_3 >$. Le crochet sur $\mathfrak{g}_0$ est égal à zéro, donc $\mathfrak{g}_0$ est une algèbre de Lie abélienne. L'action de $\mathfrak{g}_0$ sur $ZL(\mathfrak{g})$ est donnée par

$$\rho_x(y) = [(x_1, x_2, x_3, 0), (0, 0, 0, y_4)] = (0, 0, 0, 0)$$

et le 2-cocycle de Leibniz est donné par

$$\omega(x, y) = [(x_1, x_2, x_3, 0), (y_1, y_2, y_3, 0)] = (0, 0, 0, x_1y_1 + x_1y_2 - x_2y_1 + x_2y_2 + x_3y_3)$$

Maintenant, nous devons déterminer le groupe de Lie G_0 associé à $\mathfrak{g}_0$, l'action de G_0 sur $Z_L(\mathfrak{g})$ intégrant ρ et le cocycle de rack de Lie intégrant ω.

$\mathfrak{g}_0$ est une algèbre de Lie abélienne, donc un groupe de Lie intégrant $\mathfrak{g}_0$ est $G_0 = \mathfrak{g}_0$. De plus la représentation ρ est nulle, donc l'action de groupe de Lie de G_0 sur $Z_L(\mathfrak{g})$ qui intègre ρ est l'action triviale. Du fait que ρ soit nulle et $\mathfrak{g}_0$ abélienne, la différentielle $d_L^1 : CL^1(\mathfrak{g}_0, Z_L(\mathfrak{g})) \to CL^2(\mathfrak{g}_0, Z_L(\mathfrak{g}))$ est nulle. Ainis, du fait que $\mathfrak{g}_{ann} = Z_L(\mathfrak{g})$, $\mathfrak{g}$ est non scindée. Ce qu'il reste à traiter est l'intégration du cocycle ω. Une formule pour f, un cocycle de rack de Lie intégrant ω, est

$$f(a, b) = \int_{\gamma_b} (\int_{\gamma_a} \tau^2(\omega)^{eq})^{eq}$$

où $\gamma_a(s) = sa$ et $\gamma_b(t) = tb$. Soit $a \in G_0$ and $x, y \in \mathfrak{g}_0$. Nous avons

$$\begin{aligned}
\int_{\gamma_a} \tau^2(\omega)^{eq} &= \int_{[0,1]} \tau^2(\omega)^{eq}(\gamma_a(s))(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_a(s))ds \\
&= \int_{[0,1]} \Phi_{\gamma_a(s)}(\tau^2(\omega)(a))ds \\
&= \int_{[0,1]} \tau^2(\omega)(a)ds \\
&= \tau^2(\omega)(a)
\end{aligned}$$

Donc

$$\begin{aligned} f(a,b) &= \int_{\gamma_b} (\tau^2(\omega)(a))^{eq} \\ &= \int_{[0,1]} \gamma_b^*(\tau^2(\omega)(a))^{eq} \\ &= \int_{[0,1]} \tau^2(\omega)(a))^{eq}(\gamma_b(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_b(t))dt \\ &= \int_{[0,1]} \tau^2(\omega)(a))(b)dt \\ &= \tau^2(\omega)(a)(b) \\ &= \omega(a,b) \end{aligned}$$

Ainsi, la conjugaison dans $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^4$ est donnée par

$$(a_1,a_2,a_3,a_4)\rhd(b_1,b_2,b_3,b_4) = (b_1,b_2,b_3,a_1b_1+a_2b_2+a_3b_3+a_1b2-a_2b_1+b_4)$$

nous avons

$$\begin{aligned} \frac{\partial^2}{\partial s\partial t}_{|s,t=0}(sa_1,sa_2,sa_3,sa_4)\rhd(tb_1,tb_2,tb_3,tb_4) &= \frac{\partial^2}{\partial s\partial t}_{|s,t=0}(tb_1,tb_2,tb_3,st(a_1b_1+a_2b_2+a_3b_3+a_1b_2-a_2b_1),tb_4) \\ &= (0,0,0,a_1b_1+a_2b_2+a_3b_3+a_1b_2-a_2b_1) \\ &= [(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4)] \end{aligned}$$

Donc $(\mathbb{R}^4,\rhd)$ intègre $(\mathbb{R}^4,[-,-])$.

Exemple B.3.32. Soit $\mathfrak{g} = \mathbb{R}^4$. Nous définissons une application bilinéaire sur $\mathfrak{g}$ par

$$\begin{aligned} [e_1,e_1] &= e_2 \\ [e_1,e_2] &= e_3 \\ [e_1,e_3] &= e_4 \end{aligned}$$

et tous les crochets des autres combinaisons d'éléments de la base égaux à zéro. Nous avons

$$[(x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)] = (0,x_1y_1,x_1y_2,x_1y_3)$$

Proposition B.3.33. $(\mathfrak{g},[-,-])$ *est une algèbre de Leibniz.*

Preuve : Nous avons

$$\begin{aligned} [(x_1,x_2,x_3,x_4),[(y_1,y_2,y_3,y_4),(z_1,z_2,z_3,z_4)]] &= [(x_1,x_2,x_3,x_4),(0,y_1z_1,y_1z_2,y_1z_3)] \\ &= (0,0,x_1y_1z_2,x_1y_1z_3) \end{aligned}$$

et

$$[(y_1,y_2,y_3,y_4),[(x_1,x_2,x_3,x_4),(z_1,z_2,z_3,z_4)]] = (0,0,x_1y_1z_2,x_1y_1z_3)$$

et

$$[[(x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)],(z_1,z_2,z_3,z_4)] = 0$$

Ainsi le crochet $[-,-]$ satisfait l'identité de Leibniz.

□

Pour suivre la construction ci-dessus, nous devons déterminer le centre à gauche $Z_L(\mathfrak{g})$, le quotient de $\mathfrak{g}$ par $Z_L(\mathfrak{g})$ noté $\mathfrak{g}_0$, l'action de $\mathfrak{g}_0$ sur $Z_L(\mathfrak{g})$ et le 2-cocycle de Leibniz décrivant l'extension abélienne $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Soit $x \in Z_L(\mathfrak{g})$, pour $y = (1,0,0,0), y = (0,1,0,0)$ ou $y = (0,0,1,0)$ dans $\mathfrak{g}$, nous avons $[x,y] = 0$. Ceci implique que $x_1 = 0$. Réciproquement, tout élément dans $\mathfrak{g}$ avec la première coordonnée égale à 0 est dans $Z_L(\mathfrak{g})$. Ainsi $Z_L(\mathfrak{g}) =< e_2, e_3, e_4 >$ et $\mathfrak{g}_0 \simeq< e_1 >$. Le crochet sur $\mathfrak{g}_0$ est égal à zero, ainsi $\mathfrak{g}_0$ est une algèbre de Lie abélienne. L'action de $\mathfrak{g}_0$ sur $ZL(\mathfrak{g})$ est donnée par

$$\rho_x(y) = [(x_1,0,0,0),(0,y_2,y_3,y_4)] = (0,0,x_1y_2,x_1y_3)$$

et le 2-cocycle de Leibniz est donné par

$$\omega(x,y) = [(x_1,0,0,0),(y_1,0,0,0)] = (0,x_1y_1,0,0)$$

De plus, nous avons $[x,x] = (0,x_1^2,x_1x_2,x_1x_3)$, ainsi $\mathfrak{g}_{ann} = Z_L(\mathfrak{g})$. Cette algèbre de Leibniz n'est pas scindée car $\alpha \in Hom(\mathfrak{g}_0,\mathbb{R})$ et $x,y \in \mathfrak{g}_0$, nous avons $d_L\alpha(x,y) = \rho_x(\alpha(y)) = (0,0,x_1\alpha(y)_2,x_1\alpha(y)_3)$.

Maintenant, nous devons déterminer le groupe de Lie G_0 associaté à $\mathfrak{g}_0$, l'action de G_0 sur $Z_L(\mathfrak{g})$ intégrant ρ et le cocycle de rack de Lie intégrant ω.

$\mathfrak{g}_0$ est une algèbre de Lie abélienne, donc un groupe de Lie intégrant $\mathfrak{g}_0$ est $G_0 = \mathfrak{g}_0$. De plus, un simple calcul montre que l'action du groupe de Lie G_0 sur $Z_L(gg)$ définie par

$$\phi_x(y) = y + \rho_x(y)$$

intègre ρ. Il reste donc à intégrer le cocycle ω. Une formule pour f, un cocycle de rack de Lie intégrant ω, est

$$f(a,b) = \int_{\gamma_b}(\int_{\gamma_a}\tau^2(\omega)^{eq})^{eq}$$

où $\gamma_a(s) = sa$ et $\gamma_b(t) = tb$. Soit $a \in G_0$ et $x, y \in \mathfrak{g}_0$. Nous avons

$$\begin{aligned}
\int_{\gamma_a} \tau^2(\omega)^{eq} &= \int_{[0,1]} \tau^2(\omega)^{eq}(\gamma_a(s))(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_a(s))ds \\
&= \int_{[0,1]} \Phi_{\gamma_a(s)}(\tau^2(\omega)(a))ds \\
&= \int_{[0,1]} \phi_{\gamma_a(s)} \circ \tau^2(\omega)(a)ds \\
&= \int_{[0,1]} \begin{pmatrix} 0 \\ a \\ sa^2 \\ 0 \end{pmatrix} ds \\
&= \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ a^2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix}
\end{aligned}$$

Donc

$$\begin{aligned}
f(a,b) &= \int_{\gamma_b} (\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix})^{eq} \\
&= \int_{[0,1]} \gamma_b^*(\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix})^{eq}
\end{aligned}$$

$$\begin{aligned}
f(a,b) &= \int_{[0,1]} \begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix}^{eq} (\gamma_b(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_b(t))dt \\
&= \int_{[0,1]} \phi_{\gamma_b(t)}(\begin{pmatrix} 0 \\ a \\ \frac{1}{2}a^2 \\ 0 \end{pmatrix} (b))dt \\
&= \int_{[0,1]} abe_2 + (tab^2 + \frac{1}{2}a^2b)e_3 + \frac{1}{2}ta^2b^2e_4dt \\
&= abe_2 + \frac{1}{2}(ab^2 + a^2b)e_3 + \frac{1}{4}a^2b^2e_4
\end{aligned}$$

Ainsi, la conjugaison dans $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^4$ est donnée par

$$(a_1, a_2, a_3, a_4) \triangleright (b_1, b_2, b_3, b_4) = (b_1, a_1b_1, a_1b_2 + \frac{1}{2}(a_1b_1^2 + a_1^2b_1), a_1b_3 + \frac{1}{4}a_1^2b_1^2)$$

Nous avons

$$\frac{\partial^2}{\partial s \partial t}_{|s,t=0}(sa_1, sa_2, sa_3, sa_4) \rhd (tb_1, tb_2, tb_3, tb_4) = \frac{\partial^2}{\partial s \partial t}_{|s,t=0}(tb_1, sta_1b_1, sta_1b_2 + \frac{1}{2}(st^2a_1b_1^2 + s^2ta_1^2b_1),$$
$$sta_1b_3 + s^2t^2\frac{1}{4}a_1^2b_1^2)$$
$$= (0, a_1b_1, a_1b_2, a_1b_3)$$
$$= [(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)]$$

Donc $(\mathbb{R}^4, \rhd)$ intègre $(\mathbb{R}^4, [-,-])$.

En dimension 5

Exemple B.3.34. Soit $\mathfrak{g} = \mathbb{R}^5$. Nous définissons une application bilinéaire sur $\mathfrak{g}$ par

$$[e_1, e_1] = [e_1, e_2] = e_3$$
$$[e_2, e_1] = [e_2, e_2] = [e_1, e_3] = e_4$$
$$[e_1, e_4] = [e_2, e_3] = e_5$$

et tous les crochets des autres combinaisons d'éléments de la base égaux à zéro. Nous avons

$$[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)] = (0, 0, x_1(y_1 + y_2), x_2(y_1 + y_2) + x_1y_3, x_1y_4 + x_2y_3)$$

Proposition B.3.35. $(\mathfrak{g}, [-,-])$ *est une algèbre de Leibniz.*

Preuve : Nous avons

$$[(x_1, x_2, x_3, x_4), [(y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, x_1y_1(z_1+z_2), x_1(y_2(z_1+z_2)+y_1z_3)+x_2y_1(z_1+z_2))$$

et

$$[(y_1, y_2, y_3, y_4), [(x_1, x_2, x_3, x_4), (z_1, z_2, z_3, z_4)]] = (0, 0, 0, x_1y_1(z_1+z_2), x_1(y_2(z_1+z_2)+y_1z_3)+x_2y_1(z_1+z_2))$$

et

$$[[(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)], (z_1, z_2, z_3, z_4)] = (0, 0, 0, 0)$$

Ainsi le crochet $[-,-]$ satisfait l'identité de Leibniz.

$\square$

Nous devons déterminer le centre à gauche $Z_L(\mathfrak{g})$, le quotient de $\mathfrak{g}$ par $Z_L(\mathfrak{g})$ noté $\mathfrak{g}_0$, l'action de $\mathfrak{g}_0$ sur $Z_L(\mathfrak{g})$ et le 2-cocycle de Leibniz décrivant l'extension abélienne $Z_L(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0$.

Soit $x \in Z_L(\mathfrak{g})$, pour $y = (0, 0, 1, 0, 0)$ dans $\mathfrak{g}$, nous avons $[x, y] = 0$. Ceci implique que $x_1 = x_2 = 0$. Réciproquement, tout élément dans $\mathfrak{g}$ avec les deux premières coordonnées égales à 0 sont dans $Z_L(\mathfrak{g})$. Ainsi $Z_L(\mathfrak{g}) =< e_3, e_4, e_5 >$ et $\mathfrak{g}_0 \simeq < e_1, e_2 >$. Le crochet sur $\mathfrak{g}_0$ est égal à zéro, ainsi $\mathfrak{g}_0$ est une algèbre de Lie abélienne. L'action de $\mathfrak{g}_0$ sur $ZL(\mathfrak{g})$ est donnée par

$$\rho_x(y) = [(x_1, x_2, 0, 0, 0), (0, 0, y_3, y_4, y_5)] = (0, 0, 0, x_1y_3, x_1y_4 + x_2y_3)$$

et le 2-cocycle de Leibniz est donné par

$$\omega(x, y) = [(x_1, x_2, 0, 0, 0), (y_1, y_2, 0, 0, 0)] = (0, 0, x_1(y_1 + y_2), x_2(y_1 + y_2), 0)$$

De plus, nous avons $[x,x] = (0,0,x_1(x_1+x_2), x_2(x_1+x_2)+x_1x_3, x_1x_4+x_2x_3)$, ainsi en prenant $x = (1,0,0,0,0), (0,1,0,0,0)$ and $(0,1,1,0,0)$, on voit facilement que $\mathfrak{g}_{ann} = Z_L(\mathfrak{g})$. Cette algèbre de Leibniz est non scindée car pour $\alpha \in Hom(\mathfrak{g}, Z_L(\mathfrak{g}))$ et $x,y \in \mathfrak{g}_0$, nous avons $d_L\alpha(x,y) = \rho_x(\alpha(y)) = (0,0,0x_1\alpha(y)_3, x_1\alpha(y)_4 + x_2\alpha(y)_3)$.

Maintenant, nous devons déterminer le groupe de Lie G_0 associé à $\mathfrak{g}_0$, l'action de G_0 sur $Z_L(\mathfrak{g})$ intégrant ρ et le cocycle de rack de Lie intégrant ω.

L'algèbre de Lie $\mathfrak{g}_0$ est abélienne, donc un groupe de Lie intégrant $\mathfrak{g}_0$ est $G_0 = \mathfrak{g}_0$. Pour intégrer l'action ρ, nous utilisons l'exponentielle $exp : End(Z_L(\mathfrak{g})) \to Aut(Z_L(\mathfrak{g}))$. En effet, pour tout $x \in \mathfrak{g}_0$, nous avons

$$\rho_x = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ x_2 & x_1 & 0 \end{pmatrix}$$

ainsi, nous définissons un morphisme de groupe de Lie $\phi : G_0 \to Aut(Z_L(\mathfrak{g}))$ en posant

$$\phi_x = exp(\rho_x) = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 + \frac{1}{2}x_1^2 & x_1 & 0 \end{pmatrix}$$

Il est facile de voir que $d_1\phi = \rho$. Il reste donc à intégrer le cocycle ω. Une formule pour f, un cocycle de rack de Lie intégrant ω, est

$$f(a,b) = \int_{\gamma_b} (\int_{\gamma_a} \tau^2(\omega)^{eq})^{eq}$$

où $\gamma_a(s) = sa$ et $\gamma_b(t) = tb$. Soit $a \in G_0$ et $x,y \in \mathfrak{g}_0$. Nous avons

$$\begin{aligned}
\int_{\gamma_a} \tau^2(\omega)^{eq} &= \int_{[0,1]} \tau^2(\omega)^{eq}(\gamma_a(s))(\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_a(s))ds \\
&= \int_{[0,1]} \Phi_{\gamma_a(s)}(\tau^2(\omega)(a))ds \\
&= \int_{[0,1]} \phi_{\gamma_a(s)} \circ \tau^2(\omega)(a)ds \\
&= \int_{[0,1]} \begin{pmatrix} 1 & 0 & 0 \\ sa_1 & 1 & 0 \\ sa_2 + \frac{1}{2}(sa_1)^2 & sa_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_1 \\ a_2 & a_2 \\ 0 & 0 \end{pmatrix} ds \\
&= \int_{[0,1]} \begin{pmatrix} a_1 & a_1 \\ sa_1^2 + a_2 & sa_1^2 + a_2 \\ 2sa_1a_2 + \frac{1}{2}s^2a_1^3 & 2sa_1a_2 + \frac{1}{2}s^2a_1^3 \end{pmatrix} ds
\end{aligned}$$

Donc

$$\int_{\gamma_a} \tau^2(\omega)^{eq} = \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}$$

Ainsi

$$\begin{aligned}
f(a,b) &= \int_{\gamma_b} (\int_{\gamma_a} \tau^2(\omega)^{eq})^{eq} \\
&= \int_{\gamma_b} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq} \\
&= \int_{[0,1]} \gamma_b^* \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq} \\
&= \int_{[0,1]} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix}^{eq} (\gamma_b(t))(\frac{\partial}{\partial t}\Big|_{t=0} \gamma_b(t))dt \\
&= \int_{[0,1]} \phi_{\gamma_b(t)}(\begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix} (b))dt
\end{aligned}$$

Nous avons

$$\begin{aligned}
\phi_{\gamma_b(t)}(\begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix} (b)) &= \begin{pmatrix} 1 & 0 & 0 \\ tb_1 & 1 & 0 \\ tb_2 + \frac{1}{2}(tb_1)^2 & tb_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_1 \\ \frac{1}{2}a_1^2 + a_2 & \frac{1}{2}a_1^2 + a_2 \\ a_1a_2 + \frac{1}{6}a_1^3 & a_1a_2 + \frac{1}{6}a_1^3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1(b_1+b_2) \\ (tb_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1+b_2) \\ (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{2}tb_1a_1^2 + tb_2a_1 + tb_1a_2 + \frac{1}{2}(tb_1)^2a_1)(b_1+b_2) \end{pmatrix}
\end{aligned}$$

Donc

$$f(a,b) = \begin{pmatrix} a_1(b_1+b_2) \\ (\frac{1}{2}b_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1+b_2) \\ (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{4}b_1a_1^2 + \frac{1}{2}b_2a_1 + \frac{1}{2}b_1a_2 + \frac{1}{6}(b_1)^2a_1)(b_1+b_2) \end{pmatrix}$$

et la conjugaison dans $G_0 \times_f Z_L(\mathfrak{g}) = \mathbb{R}^5$ est donnée par

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} \rhd \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 + a_1(b_1+b_2) \\ a_1b_3 + b_4 + (\frac{1}{2}b_1a_1 + a_2 + \frac{1}{2}a_1^2)(b_1+b_2) \\ (a_2 + \frac{1}{2}a_1^2)b_3 + a_1b_4 + b_5 + (a_1a_2 + \frac{1}{6}a_1^3 + \frac{1}{4}b_1a_1^2 + \frac{1}{2}b_2a_1 + \frac{1}{2}b_1a_2 + \frac{1}{6}(b_1)^2a_1)(b_1+b_2) \end{pmatrix}$$

Avec un simple calcul, nous vérifions que

$$\frac{\partial^2}{\partial s \partial t}\Big|_{s,t=0} (sa_1, sa_2, sa_3, sa_4, sa_5) \rhd (tb_1, tb_2, tb_3, tb_4,\ tb_5) = [(a_1,a_2,a_3,a_4,a_5),(b_1,b_2,b_3,b_4,\ b_5)]$$

Donc $(\mathbb{R}^5, \rhd)$ intègre $(\mathbb{R}^5, [-,-])$.

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Résumé : Le résultat principal de cette thèse est une solution locale du *problème des coquecigrues.* Par problème des coquecigrues, nous parlons du problème d'intégration des algèbres de Leibniz. Cette question consiste à trouver une généralisation du troisième théorème de Lie pour les algèbres de Leibniz. Ce théorème établit que pour toute algèbre de Lie $\mathfrak{g}$, il existe un groupe de Lie G dont l'espace tangent en 1 est muni d'une structure d'algèbre de Lie isomorphe à $\mathfrak{g}$. La sructure d'algèbre de Leibniz généralise celle d'algèbre de Lie, nous cherchons donc une structure algébrique généralisant celle de groupe et répondant à la même question. Nous résolvons ce problème en intégrant localement toute algèbre de Leibniz en un *rack de Lie augmenté local.* Un rack de Lie étant une variété munie d'un produit satisfaisant plusieurs axiomes qui généralisent des propriétés de la conjugaison dans un groupe. En particulier, ce produit est autodistributif. Notre approche de ce problème est basée sur une preuve donnée par E.Cartan dans le cas des groupes et algèbres de Lie, et consiste à associer à toute algèbres de Leibniz une extension abélienne d'une algèbre de Lie par un module antisymétrique. Cette extension est caractérisée par une classe dans le second groupe de cohomologie de Leibniz, et nous associons à tous représentant de cette classe un cocyle de rack de Lie local qui nous permet de construire un rack de Lie augmenté local répondant au problème. Pour construire ce cocycle, nous généralisons une méthode d'intégration d'un cocycle d'algèbre de Lie en cocycle de groupe de Lie due à W.T.Van Est.

Summary : The main result of this thesis is a local answer to the *coquecigrue problem.* By coquecigrue problem, we mean the problem of integrating Leibniz algebras. This question consists in finding a generalization of Lie's third theorem for Leibniz algebras. This theorem establishes that for every Lie algebra $\mathfrak{g}$, there exists a Lie group G whose tangent space at 1 is provided with the structure of a Lie algebra isomorphic to $\mathfrak{g}$. The Leibniz algebra structure generalizes that of a Lie algebra, so we search for an algebraic structure generalizing that of a group and answering the same question. We answer to this problem by locally integrating every Leibniz algebra into a *local augmented Lie racks.* A Lie rack being a manifold provided with a product satisfying further axioms which generalize (some properties of) the conjugation in a group. Particularly we ask this product to be self distributive. Our approach to the problem is similar to one given by E. Cartan in the case of Lie groups and Lie algebras, and consists in associating to every Leibniz algebra an abelian extension of a Lie algebra by an anti-symmetric module. This extension is caracterised by a class in the second Leibniz cohomology group, and we associate to a representant of this class a local Lie rack cocycle which permits us to construct a local augmented Lie rack which solves the problem. To construct this local cocycle, we generalize an integration method of a Lie algebra cocycle into a Lie group cocycle due to W.T. Van Est.

Mots clés : algèbre de Leibniz, coquecigrue, rack de Lie, cohomologie d'algèbre de Leibniz, cohomologie de rack, intégration des cocycles de Leibniz.

Discipline : Mathématiques

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